CLASSIFICARE LE SINGOLARITÀ

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Come si fa la **classificazione delle singolarità di una funzione complessa**? Sto affrontando i classici esercizi di Analisi Complessa in cui, data una funzione, si richiede ti *trovare le singolarità isolate e di classificarle*.

Finché si tratta di cercare le singolarità non ho problemi, ma quando devo classificarle e dire di che tipo sono (**singolarità eliminabile**, **singolarità essenziale** o **polo**) non so come muovermi di preciso e faccio un sacco di confusione...

Forse chiedo troppo, ma l'ideale sarebbe una sorta di scaletta da poter seguire. :)

Domanda di FAQ

Soluzioni

Ciao! Vediamo di chiarire ogni dubbio. ;)

Dopo aver trovato i punti singolari procediamo in questo modo: osserviamo la funzione dritta negli occhi e facciamo subito una distinzione:

1) La funzione è facilmente sviluppabile in serie di Laurent nel punto singolare trovato

Bene! Scriviamone lo sviluppo in serie ed il gioco è fatto! Infatti:

1a) se la parte singolare dello sviluppo ha infiniti termini siamo di fronte ad una singolarità essenziale;

1b) se la parte singolare dello sviluppo ha un numero finito di termini siamo di fronte ad un polo;

1c) se la parte singolare viene a mancare, il nostro punto singolare è una singolarità di tipo eliminabile;

2) Scrivere lo sviluppo in serie di Laurent è proibitivo (accade molto spesso)

Non allarmiamoci! Ci sono un sacco di risultati che vengono in nostro aiuto. Supposto che z_0 sia il punto singolare trovato, calcoliamo:

$$\lim_{z \to z_0} |f(z)|$$

Se:

2a) Tale limite non esiste, z_0 è una singolarità di tipo essenziale

2b) Il limite esiste finito siamo di fronte ad una singolarità di tipo eliminabile

2c) Il limite è $+\infty$, z_0 è un polo.

Risposta di Galois

based on Youmath "classificare le singolarità"

Chapter 9

Isolated Singularities and the Residue Theorem

$\frac{1}{r^2}$ has a nasty singularity at $r = 0$, but it did not bother Newton—the moon is far enough.
Edward Witten

We return one last time to the starting point of Chapters 7 and 8: the quest for

$$\int_{C[2,3]} \frac{\exp(z)}{\sin(z)}\, dz \, .$$

We computed this integral in Example 8.28 crawling on hands and knees (but we finally computed it!), by considering various Taylor and Laurent expansions of $\exp(z)$ and $\frac{1}{\sin(z)}$. In this chapter, we develop a calculus for similar integral computations.

9.1 Classification of Singularities

What are the differences among the functions $\frac{\exp(z)-1}{z}$, $\frac{1}{z^4}$, and $\exp(\frac{1}{z})$ at $z = 0$? None of them are defined at 0, but each singularity is of a different nature. We will frequently consider functions in this chapter that are holomorphic in a disk except at its center (usually because that's where a singularity lies), and it will be handy to define the **punctured disk** with center z_0 and radius R,

$$\dot{D}[z_0, R] \ := \ \{z \in \mathbb{C} : \ 0 < |z - z_0| < R\} \ = \ D[z_0, R] \setminus \{z_0\} \, .$$

We extend this definition naturally with $\dot{D}[z_0, \infty] := \mathbb{C} \setminus \{z_0\}$. For complex functions there are three types of singularities, which are classified as follows.

Definition. If f is holomorphic in the punctured disk $\dot{D}[z_0, R]$ for some $R > 0$ but not at $z = z_0$, then z_0 is an **isolated singularity** of f. The singularity z_0 is called

(a) **removable** if there exists a function g holomorphic in $D[z_0, R]$ such that $f = g$ in $\dot{D}[z_0, R]$, *Proof.* (a) Suppose that z_0 is a removable singularity of f, so there exists a holomorphic function h on $D[z_0, R]$ such that $f(z) = h(z)$ for all $z \in \dot{D}[z_0, R]$. But then h is continuous at z_0, and so

$$\lim_{z \to z_0} (z - z_0) \, f(z) \ = \ \lim_{z \to z_0} (z - z_0) \, h(z) \ = \ h(z_0) \lim_{z \to z_0} (z - z_0) \ = \ 0 \, .$$

Conversely, suppose that $\lim_{z \to z_0} (z - z_0) \, f(z) = 0$ and f is holomorphic in $\dot{D}[z_0, R]$. We define the function $g : D[z_0, R] \to \mathbb{C}$ by

$$g(z) := \begin{cases} (z - z_0)^2 \, f(z) & \text{if } z \neq z_0 \, , \\ 0 & \text{if } z = z_0 \, . \end{cases}$$

Then g is holomorphic in $\dot{D}[z_0, R]$ and

$$g'(z_0) \ = \ \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} \ = \ \lim_{z \to z_0} \frac{(z - z_0)^2 \, f(z)}{z - z_0} \ = \ \lim_{z \to z_0} (z - z_0) \, f(z) \ = \ 0 \, ,$$

so g is holomorphic in $D[z_0, R]$. We can thus expand g into a power series

$$g(z) \ = \ \sum_{k \geq 0} c_k \, (z - z_0)^k$$

whose first two terms are zero: $c_0 = g(z_0) = 0$ and $c_1 = g'(z_0) = 0$. But then we can write

$$g(z) \ = \ (z - z_0)^2 \sum_{k \geq 0} c_{k+2} \, (z - z_0)^k$$

and so

$$f(z) \ = \ \sum_{k \geq 0} c_{k+2} \, (z - z_0)^k \qquad \text{for all } z \in \dot{D}[z_0, R] \, .$$

But this power series is holomorphic in $D[z_0, R]$, so z_0 is a removable singularity.

(b) Suppose that z_0 is a pole of f. Since $f(z) \to \infty$ as $z \to z_0$ we may assume that R is small enough that $f(z) \neq 0$ for $z \in \dot{D}[z_0, R]$. Then $\frac{1}{f}$ is holomorphic in $\dot{D}[z_0, R]$ and

$$\lim_{z \to z_0} \frac{1}{f(z)} \ = \ 0 \, ,$$

so part (a) implies that $\frac{1}{f}$ has a removable singularity at z_0. More precisely, the function $g : D[z_0, R] \to \mathbb{C}$ defined by

$$g(z) := \begin{cases} \frac{1}{f(z)} & \text{if } z \in \dot{D}[z_0, R] \, , \\ 0 & \text{if } z = z_0 \, , \end{cases}$$

is holomorphic. By Theorem 8.14, there exist a positive integer n and a holomorphic function h on $D[z_0, R]$ such that $h(z_0) \neq 0$ and $g(z) = (z - z_0)^n \, h(z)$. Actually, $h(z) \neq 0$ for all $z \in D[z_0, R]$ since $g(z) \neq 0$ for all $z \in \dot{D}[z_0, R]$. Thus

$$\lim_{z \to z_0} (z - z_0)^{n+1} f(z) \ = \ \lim_{z \to z_0} \frac{(z - z_0)^{n+1}}{g(z)} \ = \ \lim_{z \to z_0} \frac{z - z_0}{h(z)} \ = \ \frac{1}{h(z_0)} \lim_{z \to z_0} (z - z_0) \ = \ 0 \, .$$

Note that $\frac{1}{h}$ is holomorphic and non-zero on $D[z_0, R]$, $n > 0$, and

$$f(z) = \frac{1}{g(z)} = \frac{1}{(z - z_0)^n} \cdot \frac{1}{h(z)} \quad \text{for all } z \in \dot{D}[z_0, R]\,.$$

Conversely, suppose z_0 is not a removable singularity and $\lim_{z \to z_0} (z - z_0)^{n+1} f(z) = 0$ for some non-negative integer n. We choose the *smallest* such n. By part (a), $h(z) := (z - z_0)^n f(z)$ has a removable singularity at z_0, so there is a holomorphic function g on $D[z_0, R]$ that agrees with h on $\dot{D}[z_0, R]$. Now if $n = 0$ this just says that f has a removable singularity at z_0, which we have excluded. Hence $n > 0$. Since n was chosen as small as possible and $n - 1$ is a non-negative integer less than n, we must have $g(z_0) = \lim_{z \to z_0} (z - z_0)^n f(z) \neq 0$. Summarizing, g is holomorphic on $D[z_0, R]$ and non-zero at z_0, $n > 0$, and

$$f(z) = \frac{g(z)}{(z - z_0)^n} \quad \text{for all } z \in \dot{D}[z_0, R]\,.$$

But then z_0 is a pole of f, since

$$\lim_{z \to z_0} |f(z)| \;=\; \lim_{z \to z_0} \left| \frac{h(z)}{(z - z_0)^n} \right| \;=\; \lim_{z \to z_0} \left| \frac{g(z)}{(z - z_0)^n} \right| \;=\; |g(z_0)| \lim_{z \to z_0} \frac{1}{|z - z_0|^n} \;=\; \infty\,. \qquad \square$$

We underline one feature of the last part of our proof:

Corollary 9.6. *Suppose f is holomorphic in $\dot{D}[z_0, R]$. Then f has a pole at z_0 if and only if there exist a positive integer m and a holomorphic function $g : D[z_0, R] \to \mathbb{C}$, such that $g(z_0) \neq 0$ and*

$$\boxed{f(z) \;=\; \frac{g(z)}{(z - z_0)^m}} \quad \textit{for all } z \in \dot{D}[z_0, R]\,.$$

If z_0 is a pole then m is unique.

Proof. The only part not covered in the proof of Theorem 9.5 is uniqueness of m. Suppose $f(z) = (z - z_0)^{-m_1} g_1(z)$ and $f(z) = (z - z_0)^{-m_2} g_2(z)$ both work, with $m_2 > m_1$. Then $g_2(z) = (z - z_0)^{m_2 - m_1} g_1(z)$, and plugging in $z = z_0$ yields $g_2(z_0) = 0$, violating $g_2(z_0) \neq 0$. $\square$

Definition. The integer m in Corollary 9.6 is the **order** of the pole z_0.

This definition, naturally coming out of Corollary 9.6, parallels that of the multiplicity of a zero, which naturally came out of Theorem 8.14. The two results also show that f has a zero at z_0 of multiplicity m if and only if $\frac{1}{f}$ has a pole of order m. We will make use of the notions of zeros and poles quite extensively in this chapter.

You might have noticed that the Proposition 9.5 did not include any result on essential singularities. Not only does the next theorem make up for this but it also nicely illustrates the strangeness of essential singularities. To appreciate the following result, we suggest meditating about its statement over a good cup of coffee.

Theorem 9.7 (Casorati[1]–**Weierstraß**)**.** *If z_0 is an essential singularity of f and r is any positive real number, then every $w \in \mathbb{C}$ is arbitrarily close to a point in $f(\dot{D}[z_0, r])$. That is, for any $w \in \mathbb{C}$ and any $\epsilon > 0$ there exists $z \in \dot{D}[z_0, r]$ such that $|w - f(z)| < \epsilon$.*

[1] Felice Casorati (1835–1890).

In the language of topology, Theorem 9.7 says that the image of any punctured disk centered at an essential singularity is *dense* in $\mathbb{C}$.

There is a stronger theorem, beyond the scope of this book, which implies the Casorati–Weierstraß Theorem 9.7. It is due to Charles Emile Picard (1856–1941) and says that the image of any punctured disk centered at an essential singularity misses at most one point of $\mathbb{C}$. (It is worth coming up with examples of functions that do not miss any point in $\mathbb{C}$ and functions that miss exactly one point. Try it!)

Proof. Suppose (by way of contradiction) that there exist $w \in \mathbb{C}$ and $\epsilon > 0$ such that for all $z \in \dot{D}[z_0, r]$

$$|w - f(z)| \;\geq\; \epsilon\,.$$

Then the function $g(z) := \frac{1}{f(z)-w}$ stays bounded as $z \to z_0$, and so

$$\lim_{z\to z_0} \frac{z - z_0}{f(z) - w} \;=\; \lim_{z\to z_0} (z - z_0)\, g(z) \;=\; 0\,.$$

(Proposition 9.5(a) tells us that g has a removable singularity at z_0.) Hence

$$\lim_{z\to z_0} \left| \frac{f(z) - w}{z - z_0} \right| \;=\; \infty$$

and so the function $\frac{f(z)-w}{z-z_0}$ has a pole at z_0. By Proposition 9.5(b), there is a positive integer n so that

$$\lim_{z\to z_0} (z - z_0)^{n+1} \frac{f(z) - w}{z - z_0} \;=\; \lim_{z\to z_0} (z - z_0)^n \,(f(z) - w) \;=\; 0\,.$$

Invoking Proposition 9.5 again, we conclude that the function $f(z) - w$ has a pole or removable singularity at z_0, which implies the same holds for $f(z)$, a contradiction. $\square$

The following classifies singularities according to their Laurent series, and is very often useful in calculations.

Proposition 9.8. *Suppose z_0 is an isolated singularity of f with Laurent series*

$$f(z) \;=\; \sum_{k\in\mathbb{Z}} c_k (z - z_0)^k,$$

valid in some punctured disk centered at z_0. Then

(a) *z_0 is removable if and only if there are no negative exponents (that is, the Laurent series is a power series);*

↳ $k \geq \emptyset$ OR: $C_k = 0 \;\forall\, k < \emptyset$

(b) *z_0 is a pole if and only if there are finitely many negative exponents, and the order of the pole is the largest k such that $c_{-k} \neq 0$;*

??? ↳ $C_{-k} \neq \emptyset$ for some $k > \emptyset$

(c) *z_0 is essential if and only if there are infinitely many negative exponents.*

$\sum_{k=-\infty}^{\cdots} C_k(\cdots)$

Proof. (a) Suppose z_0 is removable. Then there exists a holomorphic function $g : D[z_0, R] \to \mathbb{C}$ that agrees with f on $\dot{D}[z_0, R]$, for some $R > 0$. By Theorem 8.8, g has a power series expansion centered at z_0, which coincides with the Laurent series of f at z_0, by Corollary 8.25.

Conversely, if the Laurent series of f at z_0 has only nonnegative powers, we can use it to define a function that is holomorphic at z_0.

(b) Suppose z_0 is a pole of order n. Then, by Corollary 9.6, $f(z) = (z - z_0)^{-n} g(z)$ on some punctured disk $\dot{D}[z_0, R]$, where g is holomorphic on $D[z_0, R]$ and $g(z_0) \neq 0$. Thus $g(z) = \sum_{k \geq 0} c_k (z - z_0)^k$ in $D[z_0, R]$ with $c_0 \neq 0$, so

$$f(z) = (z - z_0)^{-n} \sum_{k \geq 0} c_k (z - z_0)^k = \sum_{k \geq -n} c_{k+n} (z - z_0)^k \, ,$$

general example of a POLE Singularity

and this is the Laurent series of f, by Corollary 8.25.

Conversely, suppose that

$$f(z) \;=\; \sum_{k \geq -n} c_k (z - z_0)^k \;=\; (z - z_0)^{-n} \sum_{k \geq -n} c_k (z - z_0)^{k+n} \;=\; (z - z_0)^{-n} \sum_{k \geq 0} c_{k-n} (z - z_0)^k,$$

where $c_{-n} \neq 0$. Define $g(z) := \sum_{k \geq 0} c_{k-n} (z - z_0)^k$. Then g is holomorphic at z_0 and $g(z_0) = c_{-n} \neq 0$ so, by Corollary 9.6, f has a pole of order n at z_0.

(c) follows by definition: an essential singularity is neither removable nor a pole. □

Example 9.9. The order of the pole at 0 of $f(z) = \frac{\sin(z)}{z^3}$ is 2 because (by Example 8.4)

$$f(z) \;=\; \frac{\sin(z)}{z^3} \;=\; \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z^3} \;=\; \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots$$

and the smallest power of z with nonzero coefficient in this series is -2. □

9.2 Residues

We now pick up the thread from Corollary 8.27 and apply it to the Laurent series

$$f(z) \;=\; \sum_{k \in \mathbb{Z}} c_k \, (z - z_0)^k$$

at an isolated singularity z_0 of f. It says that if γ is any positively oriented, simple, closed, piecewise smooth path in the punctured disk of convergence of this Laurent series, and z_0 is inside γ, then

$$\int_\gamma f(z) \, dz \;=\; 2\pi i \, c_{-1} \, .$$

Definition. Suppose z_0 is an isolated singularity of f with Laurent series $\sum_{k \in \mathbb{Z}} c_k (z - z_0)^k$. Then c_{-1} is the **residue of** f **at** z_0, denoted by $\operatorname{Res}_{z=z_0}(f(z))$ or $\operatorname{Res}(f(z), z = z_0)$.

Corollary 8.27 suggests that we can compute integrals over closed curves by finding the residues at isolated singularities, and our next theorem makes this precise.