- Serie di Laurent

→ serie di Taylor per numeri complessi.

Laurent Series Explained | How to Determine Laurent Series | C...

Everything you need to know about Laurent Series explained. The video will contain pr...

m.youtube.com

→ The most complete example:

Example: Determine all possible Laurent Series for each function.

$$\frac{1}{1-c\cdot w} = \sum_{n=0}^{\infty} (c\cdot w)^n, \text{ for } |w| < \frac{1}{|c|} \quad (1^*)$$

$$\frac{1}{1-\frac{b}{w}} = \sum_{n=0}^{\infty} \left(\frac{b}{w}\right)^n, \text{ for } |w| > |b| \quad (2^*)$$

3) $f(z) = \dfrac{1}{(z-1)(z-2)}$, around $z=0$

$|z|<1$ $\quad$ $1<|z|<2$ $\quad$ $|z|>2$

note: $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

Partial fraction expansion

for $|z|<1$:

$$\left.\begin{aligned} &\frac{1}{z-2} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \text{ for } |z|<2 \\ &\frac{1}{z-1} = -1\cdot\frac{1}{1-z} = \{(1^*)\} = -\sum_{n=0}^{\infty} z^n, \text{ for } |z|<1 \end{aligned}\right\} f(z) = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \left(-\sum_{n=0}^{\infty} z^n\right) = \sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)\cdot z^n, \text{ for } |z|<1$$

for $1<|z|<2$:

$$\left.\begin{aligned} &\frac{1}{z-2} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \text{ for } |z|<2 \\ &\frac{1}{z-1} = \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} = \{(2^*)\} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=1}^{\infty}\frac{1}{z^n}, \text{ for } |z|>1 \end{aligned}\right\} f(z) = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=1}^{\infty}\frac{1}{z^n}, \text{ for } 1<|z|<2$$

- Wake-up:
- Definition of Laurent Series:

$$f(z) = \dots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)^1} + a_0 + a_1\cdot(z-z_0)^1 + a_2\cdot(z-z_0)^2 + a_3\cdot(z-z_0)^3 + \dots$$ Laurent Series

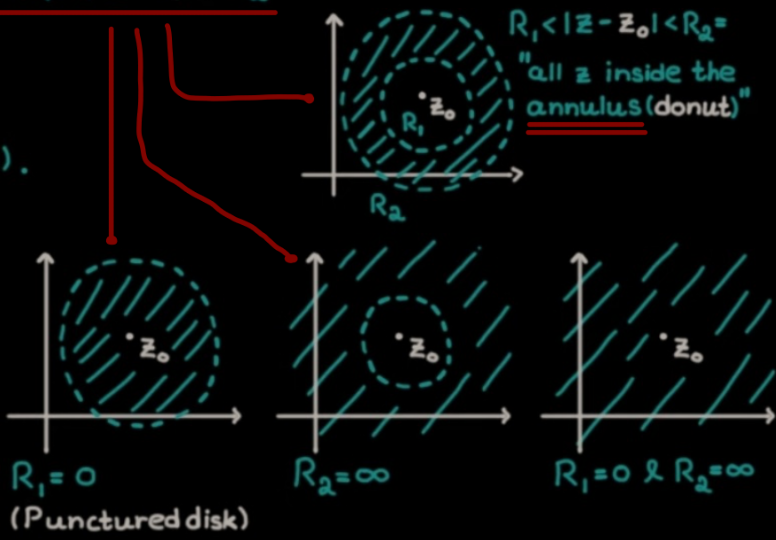
- Such that:

Theorem: Laurent Series

if $f(z)$ is analytic on $R_1 < |z-z_0| < R_2$, then

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n\cdot(z-z_0)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n}\cdot(z-z_0)^{-n}}_{\text{principal part}}, \text{ for all } z \text{ on } \underline{R_1 < |z-z_0| < R_2}$$

and where $a_n = \frac{1}{2\pi i}\oint_C \frac{f(s)}{(s-z_0)^{n+1}}\,ds \quad (n = 0, \pm 1, \pm 2, \dots)$.



- Usually we don't use the formula for calculating a_n

- We can more easily use the armonich series:

$$\frac{1}{1-w} = w^0 + w^1 + w^2 + \dots = \sum_{n=0}^{\infty} (w)^n, \text{ for } |w| < 1 \quad (1)$$

$$\frac{1}{1-\frac{1}{w}} = \frac{1^0}{w^0} + \frac{1^1}{w^1} + \frac{1^2}{w^2} + \dots = \sum_{n=0}^{\infty} \left(\frac{1}{w}\right)^n, \text{ for } |w| > 1 \quad (2)$$

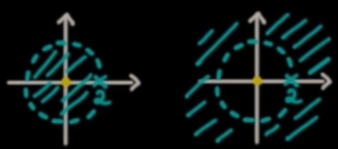
- So we can solve the following problem:

Example: Determine all possible Laurent Series for each function.

$$\frac{1}{1-c\cdot w} = \sum_{n=0}^{\infty} (c\cdot w)^n, \text{ for } |w| < \frac{1}{|c|} \quad (1^*)$$

$$\frac{1}{1-\frac{b}{w}} = \sum_{n=0}^{\infty} \left(\frac{b}{w}\right)^n, \text{ for } |w| > |b| \quad (2^*)$$

1) $f(z) = \frac{1}{z-2}$, around $z = 0$



$|z| < 2$ $\quad$ $|z| > 2$

for $|z| < 2$:

$$f(z) = \frac{1}{z-2} = -1 \cdot \frac{1}{2-z} = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = \left\{(1^*), w = z, c = \tfrac{1}{2}\right\} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n =$$

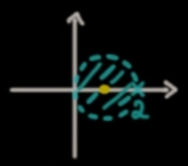
$$= -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} , \text{ for } |z| < 2$$

for $|z| > 2$:

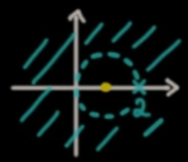
$$f(z) = \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\frac{2}{z}} = \left\{(2^*), w = z, b = 2\right\} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n =$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \{k = n+1\} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{z^n} , \text{ for } |z| > 2$$

2) $f(z)=\frac{1}{z-2}$, around $z=1$



$|z-1|<1$



$|z-1|>1$

for $|z-1|<1$:

$$f(z)=\frac{1}{z-2}=\frac{1}{(z-1)-1}=-1\cdot\frac{1}{1-(z-1)}=\{(1^*)\}=-\sum_{n=0}^{\infty}(z-1)^n,\ \text{for } |z-1|<1$$

for $|z-1|>1$:

$$f(z)=\frac{1}{z-2}=\frac{1}{(z-1)-1}=\frac{1}{z-1}\cdot\frac{1}{1-\frac{1}{z-1}}=\{(2^*)\}=\frac{1}{z-1}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n=$$

$$=\sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+1}}=\sum_{n=1}^{\infty}\frac{1}{(z-1)^n}$$

• Or another problem as:

Example: Determine all possible Laurent Series for each function.

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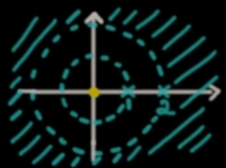
$$\frac{1}{1-\frac{b}{w}}=\sum_{n=0}^{\infty}\left(\frac{b}{w}\right)^n,\ \text{for } |w|>|b|\quad (2^*)$$

3) $f(z)=\frac{1}{(z-1)(z-2)}$, around $z=0$

$|z|<1$

$1<|z|<2$



$|z|>2$

note: $\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}$

Partial fraction expansion

for $|z|<1$:

$$\left.\begin{aligned}&\frac{1}{z-2}=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}},\ \text{for } |z|<2\\ &\frac{1}{z-1}=-1\cdot\frac{1}{1-z}=\{(1^*)\}=-\sum_{n=0}^{\infty}z^n,\ \text{for } |z|<1\end{aligned}\right\} f(z)=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}-\left(-\sum_{n=0}^{\infty}z^n\right)=\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)\cdot z^n,\ \text{for } |z|<1$$

for $1<|z|<2$:

$$\left.\begin{aligned}&\frac{1}{z-2}=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}},\ \text{for } |z|<2\\ &\frac{1}{z-1}=\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}=\{(2^*)\}=\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n=\sum_{n=1}^{\infty}\frac{1}{z^n},\ \text{for } |z|>1\end{aligned}\right\} f(z)=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}-\sum_{n=1}^{\infty}\frac{1}{z^n},\ \text{for } 1<|z|<2$$

→ So our objective is to obtain the following formula format:

$$f(z) = \sum_{m=\emptyset}^{+\infty} a_m \cdot (z-z_0)^m + \sum_{m=1}^{+\infty} \frac{a_{-m}}{(z-z_0)^m}$$

• Things to NOTE:

• $$f(z) = \underbrace{\sum_{m=\emptyset}^{+\infty} a_m \cdot (z-z_0)^m}_{\text{analytic part}} + \underbrace{\sum_{m=1}^{+\infty} \frac{a_{-m}}{(z-z_0)^m}}_{\text{principal part}}$$

• $$f(z) = \sum_{m=\emptyset}^{+\infty} a_m \cdot (z-z_0)^m + \sum_{m=1}^{+\infty} \frac{a_{-m}}{(z-z_0)^m}$$

starts at $\emptyset$

starts at 1