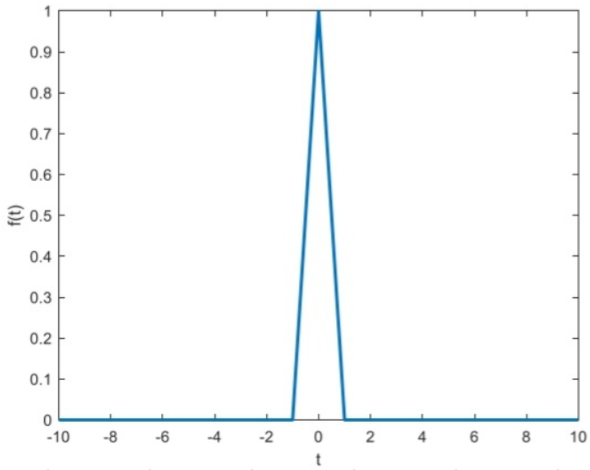
3) Consider the real function $f(t)=\begin{cases}1-|t| & |t|\le 1\\ 0 & |t|>1\end{cases}$ of a real variable $t\in\mathbb{R}$.

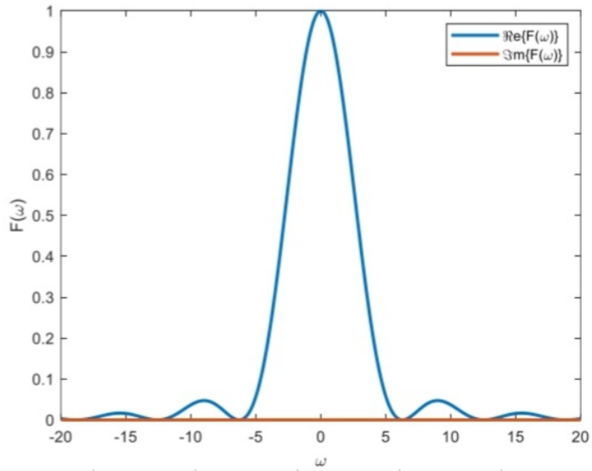
3.1 Say if and why $f(t)$ is Fourier transformable and determine $F(\omega)$ the Fourier transform of the function $f(t)$.

$\boxed{\int_{-\infty}^{+\infty}|f(t)|\,dt}=\int_{-1}^{+1}(1-|t|)\,dt=2\int_{0}^{+1}(1-t)\,dt\boxed{=1<\infty}$, so $f(t)\in L^1$ and it is Fourier transformable. The Fourier transform is given by

$$F(\omega)=\int_{-\infty}^{+\infty}f(t)e^{-i\omega t}dt=\int_{-1}^{0}f(t)e^{-i\omega t}dt+\int_{0}^{1}f(t)e^{-i\omega t}dt=2\int_{0}^{1}(1-t)\cos(\omega t)\,dt=2\int_{0}^{1}(1-t)\frac{d}{dt}\frac{\sin(\omega t)}{\omega}dt$$

$$=2\frac{(1-t)\sin(\omega t)}{\omega}\bigg|_0^1+2\int_0^1\frac{\sin(\omega t)}{\omega}dt=-2\frac{\cos(\omega t)}{\omega^2}\bigg|_0^1=2\frac{1-\cos(\omega)}{\omega^2}=\left[\frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}\right]^2$$





$\rightarrow$ If: $\displaystyle\int_{-\infty}^{+\infty}|f(t)|\,dt<\infty$

$\Rightarrow f(t)\in\mathcal{L}^1(\mathbb{R})$

$\Rightarrow f(t)$ is Fourier Transformable

→ What if $\int_{-\infty}^{+\infty} |f(t)|\,dt = \infty$ → ???

→ Probability : $f(t)$ is <u>NOT</u> Fourier transf. (maybe)

• Other examples:

3 Consider the real function $f(t)=\begin{cases}1-t^2 & |t|\le 1\\ 0 & |t|>1\end{cases}$ of a real variable $t\in\mathbb{R}$.

3.1 Say if and why $f(t)$ Fourier transformable and determine $F(\omega)$ the Fourier transform of the function $f(t)$.

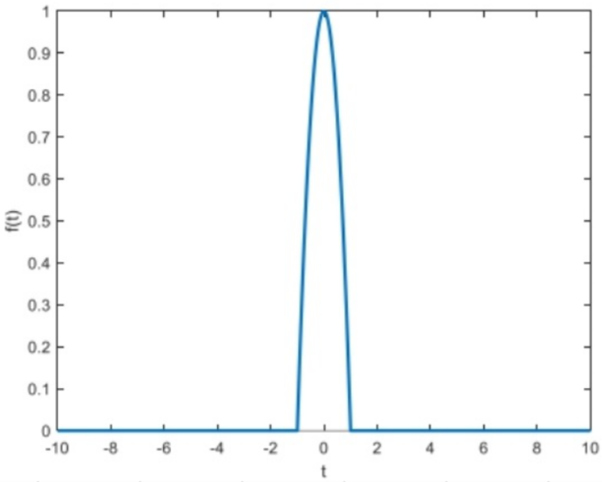
$\boxed{\int_{-\infty}^{\infty}|f(t)|dt} = \int_{-1}^{1}(1-t^2)dt = \left[t-\frac{t^3}{3}\right]_{-1}^{1} = 2-\frac{2}{3} = \boxed{\frac{4}{3}}$, hence $f(t)\in L^1(\mathbb{R})$ and $f(t)$ is Fourier transformable.

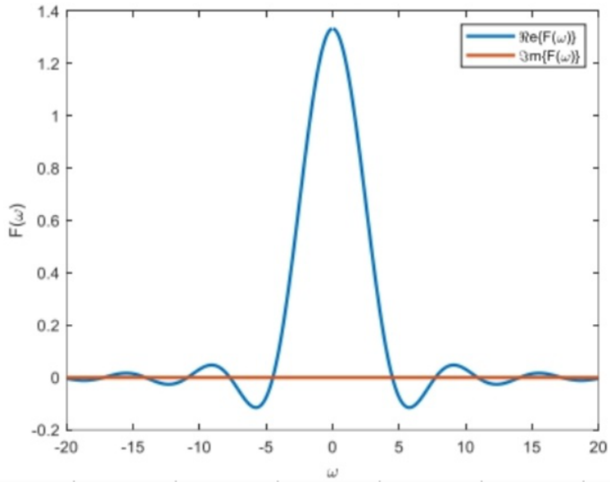
$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt=\int_{-1}^{1}(1-t^2)e^{i\omega t}dt=\int_{-1}^{1}(1-t^2)\frac{1}{i\omega}\frac{d}{dt}e^{i\omega t}dt$$

$$=\cancel{\frac{1}{i\omega}\left[(1-t^2)e^{i\omega t}\right]_{-1}^{1}}+\frac{2}{i\omega}\int_{-1}^{1}te^{i\omega t}dt=\frac{2}{i\omega}\int_{-1}^{1}t\frac{1}{i\omega}\frac{d}{dt}e^{i\omega t}dt=$$

$$=-\frac{2}{\omega^2}\left[te^{i\omega t}\right]_{-1}^{1}+\frac{2}{\omega^2}\int_{-1}^{1}e^{i\omega t}dt=-\frac{4\cos\omega}{\omega^2}+\frac{2}{\omega^2}\left[\frac{e^{i\omega t}}{i\omega}\right]_{-1}^{1}=-\frac{4\cos\omega}{\omega^2}+\frac{4\sin\omega}{\omega^3}=4\frac{\sin\omega-\omega\cos\omega}{\omega^3}$$

$$F(\omega)=\begin{cases}4\dfrac{\sin\omega-\omega\cos\omega}{\omega^3} & \omega\neq 0\\[2mm] \dfrac{4}{3} & \omega=0\end{cases}$$





3) Consider the time signal $f(t) = e^{-\frac{t^2}{2T^2}}$ (Gaussian pulse)

3.1 Calculate, if possible, the Fourier transform $F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$ of the given signal.

$f(t) \in L^2$ and it is Fourier transformable.

$F(\omega) = \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2T^2}} e^{-i\omega t} dt = e^{-\frac{T^2\omega^2}{2}} \int_{-\infty}^{+\infty} e^{-\left(\frac{t}{\sqrt{2}T} + i\frac{T\omega}{\sqrt{2}}\right)^2} dt = \sqrt{2} T e^{-\frac{T^2\omega^2}{2}} \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{2\pi} T e^{-\frac{T^2\omega^2}{2}}$. The spectrum is also a Gaussian.

3.2 Calculate the pulse duration defined as $\Delta_t = \int_{-\infty}^{+\infty} f(t) dt$ and the spectrum bandwidth defined as $\boxed{\Delta_\omega = \int_{-\infty}^{+\infty} \frac{F(\omega)}{F(0)} d\omega.}$

THIS IS A POINT
(not a multiplication)

$\boxed{\Delta_t = \int_{-\infty}^{+\infty} f(t) dt = F(0) = \sqrt{2\pi} T}$; $\boxed{\Delta_\omega = \int_{-\infty}^{+\infty} \frac{F(\omega)}{F(0)} d\omega = \frac{2\pi}{F(0)} f(0) = \frac{\sqrt{2\pi}}{T}}$. The duration-bandwidth product is constant $\underline{\Delta_t \Delta_\omega = \sqrt{2\pi} T \frac{\sqrt{2\pi}}{T}} = 2\pi$, as expected; i.e., the shorter the pulse, the narrower the spectrum, and vice versa.

$$\int_{-\infty}^{+\infty} f(t)\, dt = F(\omega = 0)$$

$$\int_{-\infty}^{+\infty} F(\omega)\, d\omega = f(t = 0)$$

PROPERTY of the Fourier Transform