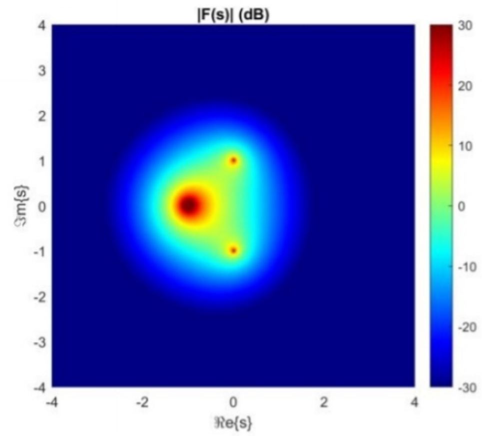
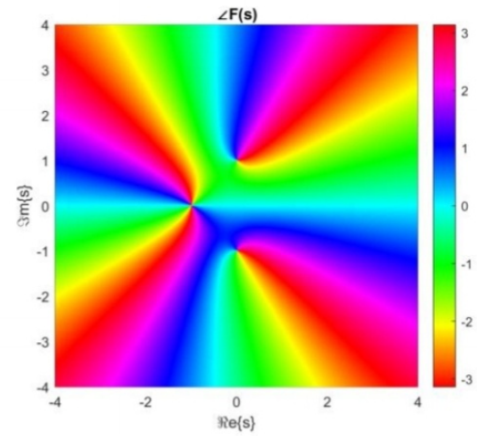
4) Solve the following second order Cauchy problem by resorting to Laplace transform by finding the solution $f(t)$ for $t \geq 0$.

$$\begin{cases} \dfrac{d^2}{dt^2} f(t) + 2\dfrac{d}{dt} f(t) + f(t) = \sin(t) \\ f(0) = 0 \\ \dfrac{d}{dt} f(0) = 0 \end{cases}$$

By applying the Laplace transform to the differential equation one obtains $\left(s^2+2s+1\right)F(s) = \dfrac{1}{s^2+1}$,

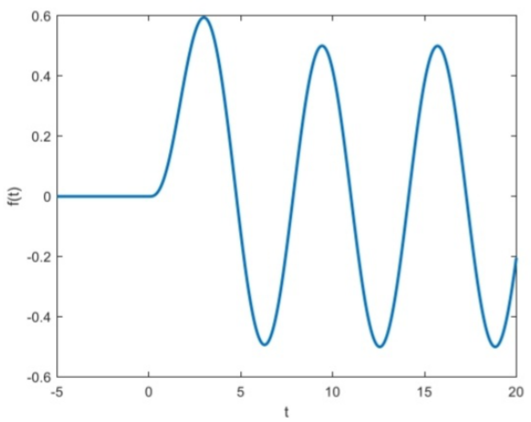
where $F(s) = \mathcal{L}\{f(t)\}$, whence $F(s) = \dfrac{1}{\left(s^2+1\right)\left(s+1\right)^2}$.




Then, the solution is found via inverse Laplace transform as $f(t) = \dfrac{1}{2\pi i}\displaystyle\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds$, with $\sigma = 0^+$ being the abscissa of convergence. The integral is calculated by closing the path at infinity by invoking the Jordan's Lemma on the right $\mathrm{Re}\{s\} > 0$ $t<0$ and obtaining $f(t) = 0$ as the integrand is holomorphic inside the integration path. Conversely, when $t > 0$ the integration path is closed at infinity on the left $\mathrm{Re}\{s\} < 0$ and can be calculated as the sum of the residues at the poles $s = \pm i$ and $s = -1$, i.e.,

$$\boxed{\begin{aligned} f(t) &= \mathrm{Res}\left\{F(s)e^{st}, s=i\right\} + \mathrm{Res}\left\{F(s)e^{st}, s=-i\right\} + \mathrm{Res}\left\{F(s)e^{st}, s=-1\right\} \\ &= \lim_{s\to i}(s-i)F(s)e^{st} + \lim_{s\to -i}(s+i)F(s)e^{st} + \lim_{s\to -1}\frac{d}{ds}(s+1)^2 F(s)e^{st} \\ &= -\frac{e^{it}}{4} - \frac{e^{-it}}{4} + \frac{1+t}{2}e^{-t} = -\frac{1}{2}\cos(t) + \frac{1+t}{2}e^{-t} \end{aligned}}$$



• Jordan's Lemma:

**Theorem 4.27** (Cauchy's Integral Formula). *Suppose $f$ is holomorphic in the region $G$ and $\gamma$ is a positively oriented, simple, closed, piecewise smooth path, such that $w$ is inside $\gamma$ and $\gamma \sim_G 0$. Then*

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w}\, dz\,.$$

**Example 4.29.** Continuing Example 4.26, Theorem 4.27 says that

$$\int_\gamma \frac{dz}{z^2+1} = \pi$$

for any positively oriented, simple, closed, piecewise smooth path $\gamma$ that contains $i$ on its inside and that is $(\mathbb{C} \setminus \{-i\})$-contractible. □

**Example 4.30.** To compute

$$\int_{C[0,3]} \frac{\exp(z)}{z^2-2z}\, dz$$

we use the partial fractions expansion from Example 4.23:

$$\int_{C[0,3]} \frac{\exp(z)}{z^2-2z}\, dz = \frac{1}{2}\int_{C[0,3]} \frac{\exp(z)}{z-2}\, dz - \frac{1}{2}\int_{C[0,3]} \frac{\exp(z)}{z}\, dz\,.$$

For the two integrals on the right-hand side, we can use Theorem 4.24 with the function $f(z) = \exp(z)$, which is entire, and so (note that both 2 and 0 are inside $\gamma$)

$$\int_{C[0,3]} \frac{\exp(z)}{z^2-2z}\, dz = \frac{1}{2}2\pi i \cdot \exp(2) - \frac{1}{2}2\pi i \cdot \exp(0) = \pi i\left(e^2-1\right).$$ □

N.B.: In the book the only mention of Jordan is when defining the interior and outside part of a closed curve and its orientation (positive oriented → clockwise)

- Another usefull piece of knowledge to solve the integrals are the Residual (Integration for holomorphic functions)