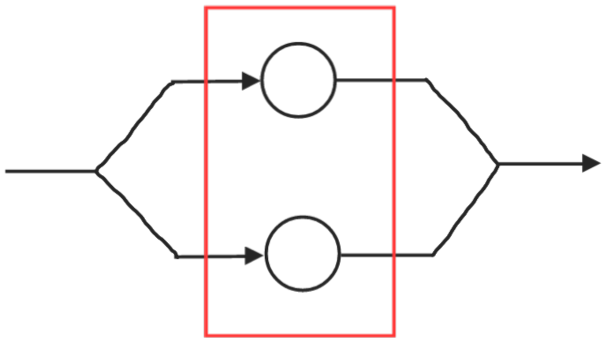
Exercise 2

A low-cost hotel has a small fitness centre with only two identical equipments. A guest willing to have physical activity, uses one of the equipments, if available, otherwise she comes back to her room. The fitness centre opens at 10 AM and closes at 10 PM. The guests arrive at the fitness centre according to a Poisson process with average interarrival time equal to 15 min, whereas the duration of the guest's use of an equipment has a distribution which can be approximated by an exponential distribution with expected value 30 min, independent of the guest.

1. Model the fitness centre using a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, f, x_0, F)$.
2. Compute the probability that both equipments are busy at 10:20 AM.
3. Compute the probability that at least one guest has to stop training at 10 PM.

The system can be represented as two identical, parallel servers with no queue. The servers are the equipments.



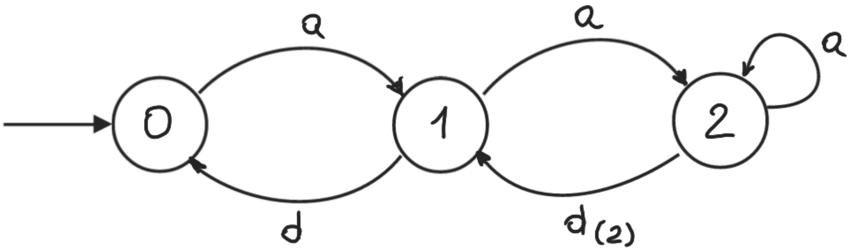
Fitness centre

1. Events $\mathcal{E} = \{ a \, , \, d \}$

a: Arrival of a guest

d: Termination of the use of an equipment

State x = # guests in the fitness centre $\in \{0, 1, 2\}$



$p_{x_0}(0) = 1, \quad p_{x_0}(1) = p_{x_0}(2) = 0$

$F = \{F_a, F_d\}$

$F_a(t) = 1 - e^{-\lambda t}, \quad t \geq 0 \qquad \frac{1}{\lambda} = 15 \text{ min} = \frac{1}{4} \text{ hours}$

$\Rightarrow \lambda = 4$ arrivals/hour

$F_d(t) = 1 - e^{-\mu t}, \quad t \geq 0 \qquad \frac{1}{\mu} = 30 \text{ min} = \frac{1}{2} \text{ hours}$

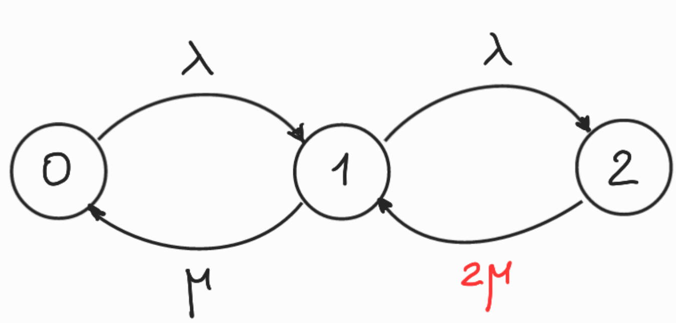
$\Rightarrow \mu = 2$ services/hour

$\Downarrow$

Poisson clock structure

2. We have to compute $P(X(t)=2)$, where $t = 20$ min $= \frac{1}{3}$ hours

To accomplish this, we first transform the stochastic timed automaton with Poisson clock structure into an equivalent CTHMC.



$$\Rightarrow Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{bmatrix} = \begin{bmatrix} -4 & 4 & 0 \\ 2 & -6 & 4 \\ 0 & 4 & -4 \end{bmatrix}$$

$$\Pi_0 = [\underbrace{1}_{p_{x_0}(0)} \quad \underbrace{0}_{p_{x_0}(1)} \quad \underbrace{0}_{p_{x_0}(2)}]$$

$$\Rightarrow P(\underbrace{X(t)}_{t=\frac{1}{3}\text{ hours}}=2) = \Pi(t)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \Pi_0 e^{Qt}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \simeq 0.2338$$

$$\Pi(t) = [P(X(t)=0) \; P(X(t)=1) \; P(X(t)=2)]$$

Using Matlab:

```
% Matrix Q
Q = [ -4 4 0 ; 2 -6 4 ; 0 4 -4 ];

% Initial state probability vector
pi0 = [ 1 0 0 ];

% State probability vector at time t
t = 1/3;
pit = pi0*expm(Q*t)
```

Result:

```
pit =

    0.3805    0.3857    0.2338
```

3. We have to compute $P(X(t)=1)+P(X(t)=2)$,

where $t = 12$ hours.

With the same approach as before:

$$P(X(t)=1)+P(X(t)=2) = \pi(t)\begin{bmatrix}0\\1\\1\end{bmatrix} = \pi_0 e^{Qt}\begin{bmatrix}0\\1\\1\end{bmatrix} \simeq 0.800$$

Here $t = 12$ hours

Using Matlab:

```
% State probability vector at time t
t = 12;
pit = pi0*expm(Q*t)
```

Result:

```
pit =

    0.2000    0.4000    0.4000
```

REMARK

At time $t = 12$ hours, the system can be considered in practice at steady state.

This can be observed plotting the state probabilities versus time.

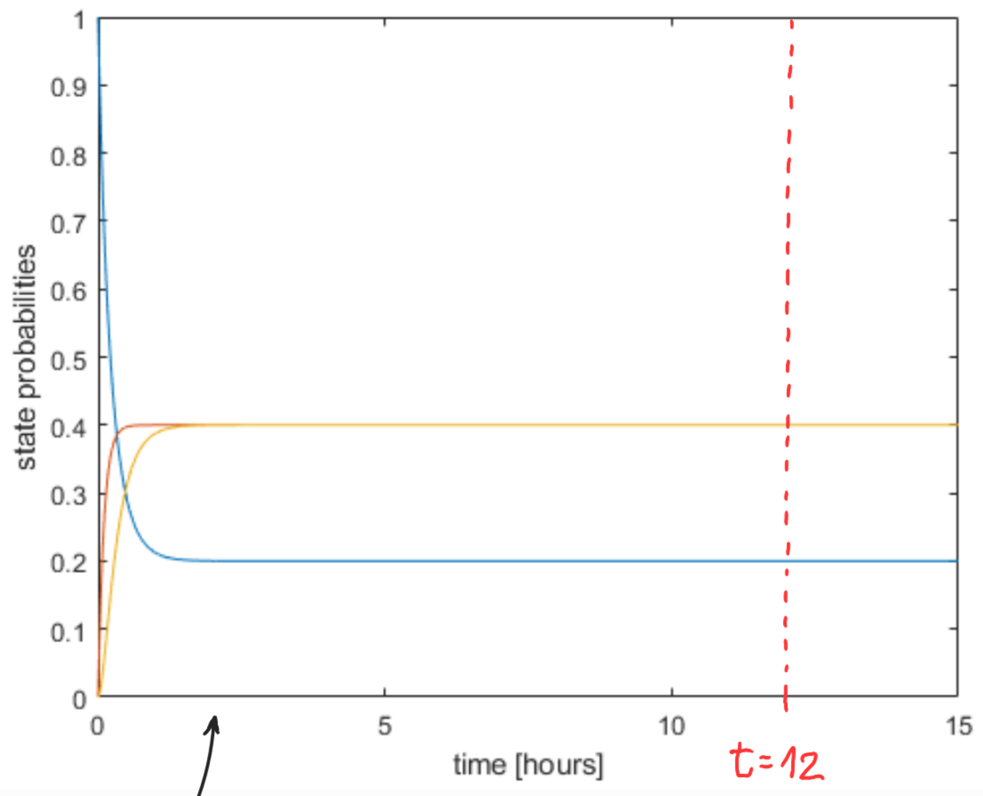
Using Matlab:

```
% Matrix Q
Q = [ -4 4 0 ; 2 -6 4 ; 0 4 -4 ];

% Initial state probability vector
pi0 = [ 1 0 0 ];

% Plot of state probabilities vs time
T = 0:0.01:15; % Time grid
PI = [];
for t = T,
    PI(end+1,:) = pi0*expm(Q*t); % State probability vector at time t
end
plot(T,PI)
xlabel('time [hours]')
ylabel('state probabilities')
```

Plot:



Notice that the system is in practice at steady state already after two hours

Hence, point 3) could be tackled replacing state probabilities at time $t=12$ hours with limit probabilities (the approximation error doing this is negligible).

The CTHMC is irreducible and finite.

=> Limit probabilities can be obtained solving the linear system of equations:

$$\begin{cases} \pi Q = 0 \\ \sum_i \pi_i = 1 \end{cases} \Rightarrow \begin{cases} -4\pi_1 + 2\pi_2 = 0 \\ \cancel{4\pi_1 - 6\pi_2 + 4\pi_3 = 0} \quad \text{redundant} \\ 4\pi_2 - 4\pi_3 = 0 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$$\begin{cases} \pi_1 = \frac{1}{2}\pi_2 & \Rightarrow \quad \pi_1 = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5} \\ \pi_3 = \pi_2 & \Rightarrow \quad \pi_3 = \frac{2}{5} \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$$\frac{1}{2}\pi_2 + \pi_2 + \pi_2 = 1 \Rightarrow \frac{5}{2}\pi_2 = 1 \Rightarrow \pi_2 = \frac{2}{5}$$

$$\Rightarrow \pi = [\pi_1 \ \pi_2 \ \pi_3] = \left[\frac{1}{5} \ \frac{2}{5} \ \frac{2}{5}\right] = [0.2 \ 0.4 \ 0.4]$$

Compare it with $\pi(t)$ computed for $t = 12$ hours