Exercise 1

The alarm system of a building is equipped with a keypad located inside the entrance door. Assume that the alarm is active, and let time $t = 0$ be the instant when the entrance door is opened. A security code of three digits must be entered to deactivate the alarm using the keypad. At time $t = 0$, a timer T_1 of 15 s starts. If the correct code is not entered before T_1 expires, the alarm goes off. Moreover, another timer T_2 of 5 s sets the maximum time to enter a new digit. If a new digit is not entered before T_2 expires, the system is reset to the initial state, waiting for the complete sequence. The same occurs when a wrong digit is entered in the sequence. Timer T_2 starts the first time at time $t = 0$, and then restarts every time a digit is entered, or after it expires. For the sake of simplicity, assume that, when the alarm either goes off or is deactivated, the system enters states where the timers are not active, while entering any digit does not cause any effect.

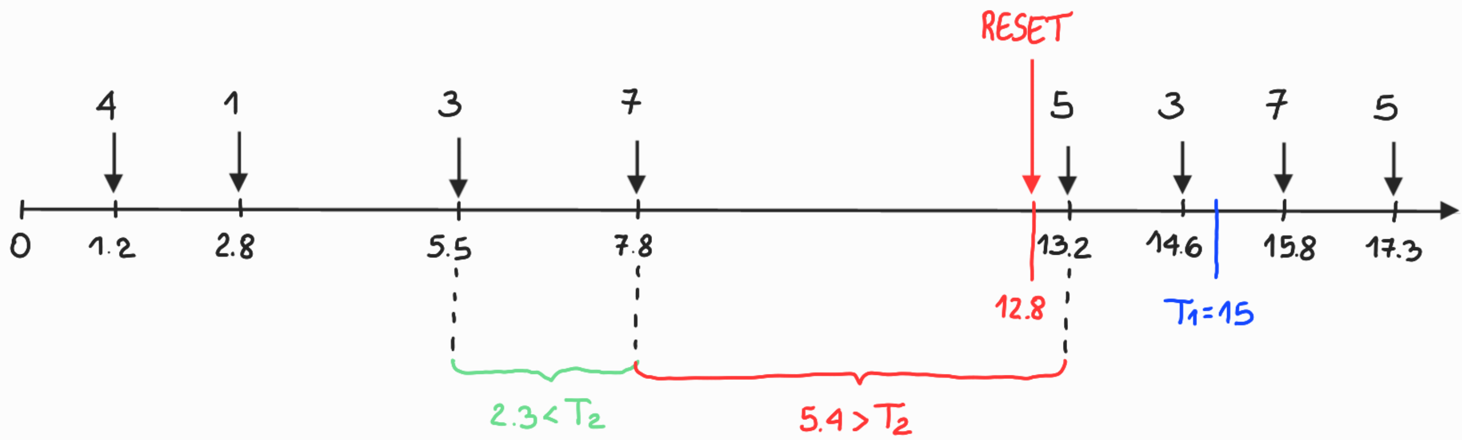
1. Assume that the security code is 375, and that the sequence 41375375 is entered, with the eight digits entered at times 1.2, 2.8, 5.5, 7.8, 13.2, 14.6, 15.8, and 17.3. Build the sample path of the system. Does the alarm go off, or is it deactivated?

Now assume that the time the user takes to think and enter a digit, has a uniform distribution over the interval $[3, 8]$ s. Moreover, let $q = 3/4$ be the probability that an entered digit is the correct one in the sequence.

2. Model the system using a stochastic timed automaton $\{\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F\}$.
3. Compute the probability that the user deactivates the alarm entering only three digits.
4. Compute the probability that the user deactivates the alarm.

1. To draw the sample path, we can avoid defining a model first. While drawing it, just keep in mind that:

- if the time interval between consecutive digits is greater than $T_2 = 5\,s$, the system is reset (previously entered digits are forgotten $\Rightarrow$ a new sequence must be started);
- if the correct sequence is not entered by $T_1 = 15\,s$ after opening the door, the alarm goes off.



$\Downarrow$

The correct subsequence 375 is interrupted by a reset

$\Downarrow$

The alarm goes off

2. Stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$

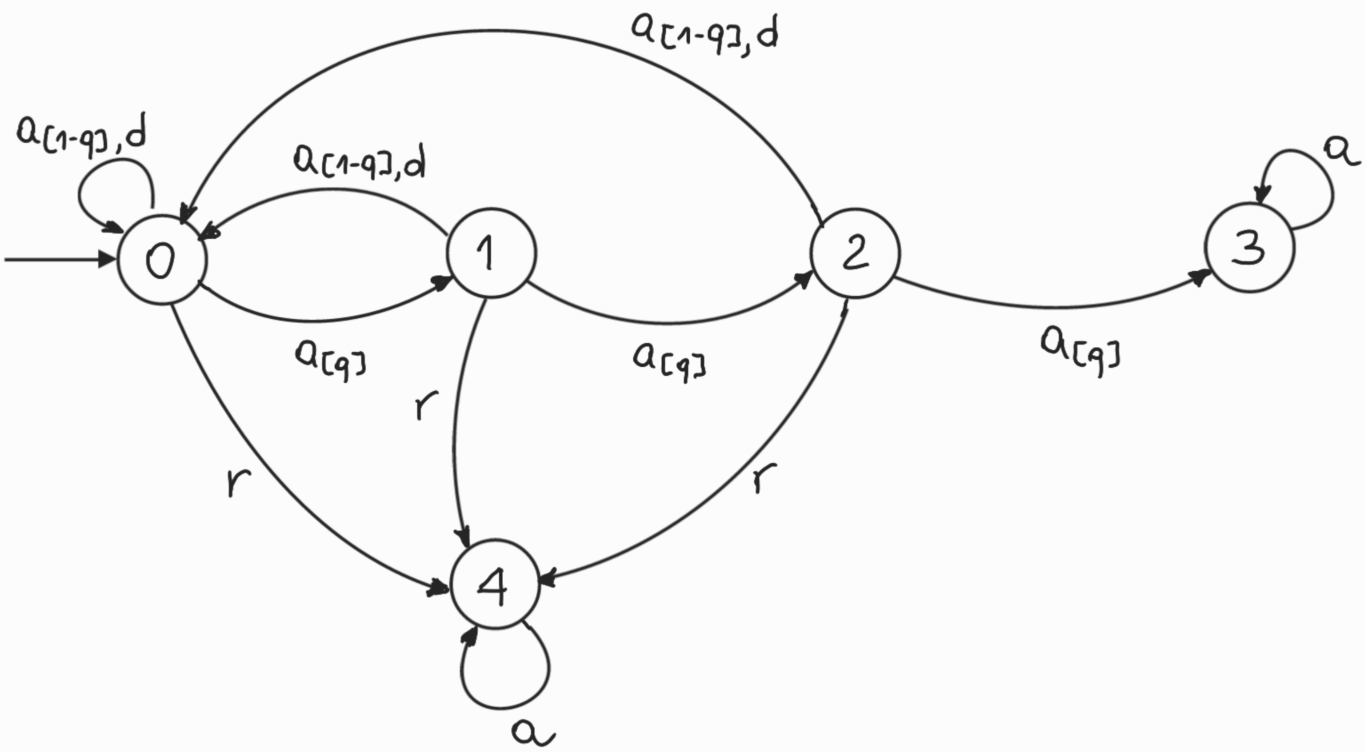
Events $\mathcal{E} = \{a, d, r\}$

A digit is entered

Timer T_2 expires

Timer T_1 expires

State $x = \begin{cases} 0 & : \text{reset state} \\ 1 & : \text{one correct digit entered so far} \\ 2 & : \text{two correct digits entered so far} \\ 3 & : \text{alarm deactivated} \\ 4 & : \text{alarm gone off} \end{cases}$



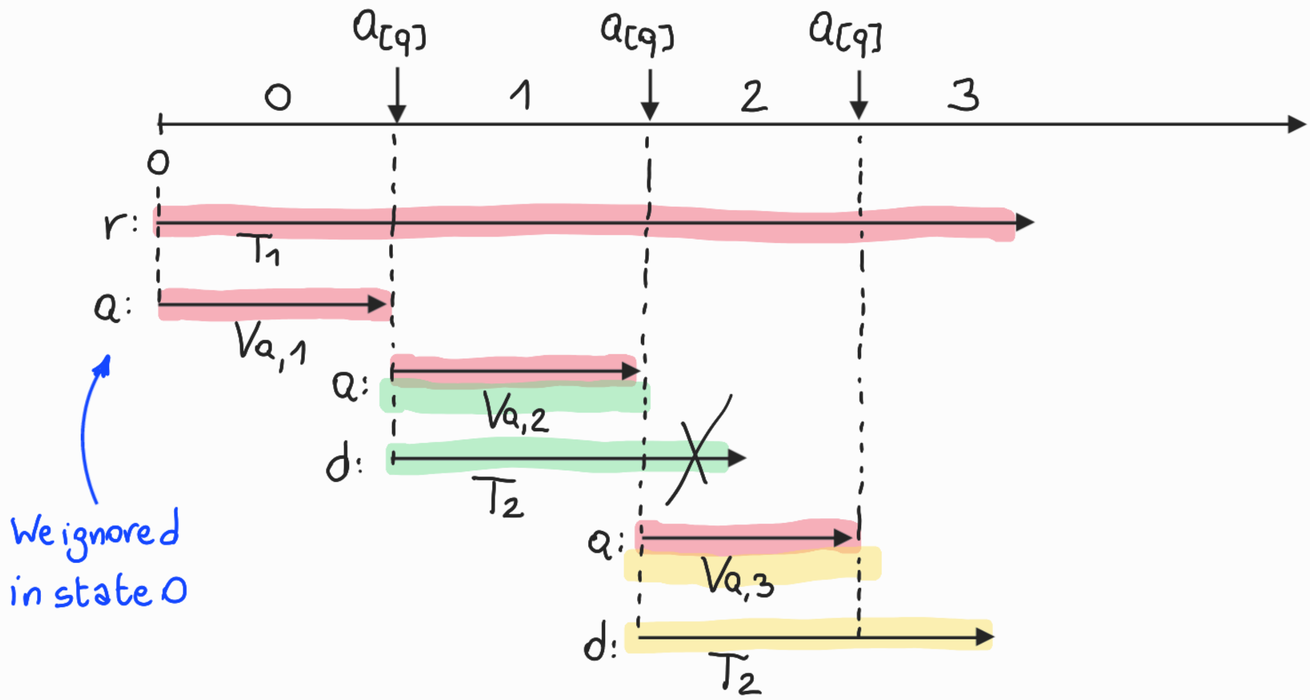
Stochastic clock structure:

$$\left.\begin{aligned} V_a &\sim U(3,8)\ s \\ V_d &= T_2 = 5\ s \\ V_r &= T_1 = 15\ s \end{aligned}\right\} \text{mix of deterministic and stochastic lifetimes}$$

3. Only one case:

$$0 \xrightarrow[\text{ignored}]{a[q]} 1 \xrightarrow{a[q]} 2 \xrightarrow{a[q]} 3$$

Sample path:



$$\Rightarrow P(\ldots) = q^3 P\left(V_{a,2} < T_2,\ V_{a,3} < T_2,\ V_{a,1} + V_{a,2} + V_{a,3} < T_1\right)$$

4. Notice that, since $V_a \sim U(3,8)$ s, the user can enter a maximum of 4 digits over the interval $[0,15)$ s.

$\Rightarrow$ We can restrict the attention to paths from state 0 to state 3 involving a maximum of 4 events a.

Three cases:

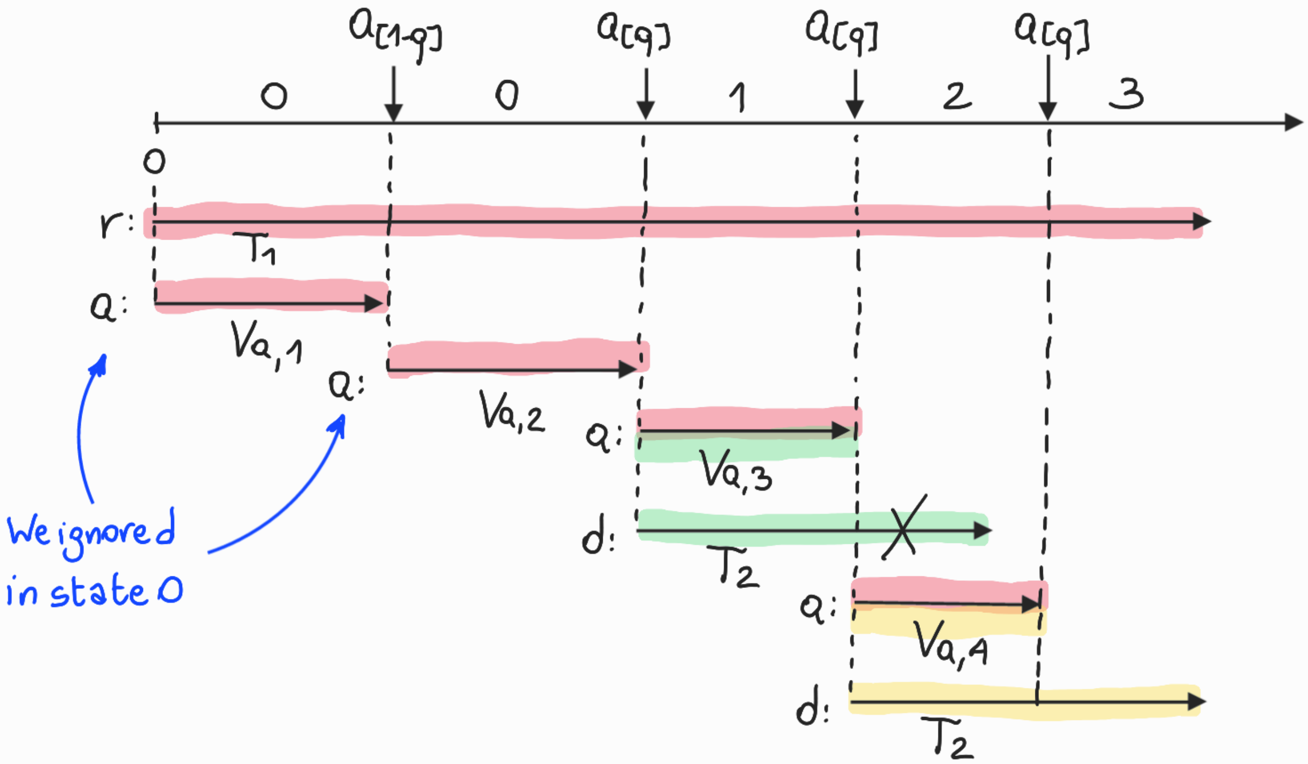
$\boxed{1}$ $\quad 0 \underset{\text{ignored}}{\xrightarrow{a_{[q]}}} 1 \xrightarrow{a_{[q]}} 2 \xrightarrow{a_{[q]}} 3$

$\boxed{2}$ $\quad 0 \underset{\text{ignored}}{\xrightarrow{a_{[1-q]}}} 0 \underset{\text{ignored}}{\xrightarrow{a_{[q]}}} 1 \xrightarrow{a_{[q]}} 2 \xrightarrow{a_{[q]}} 3$

$\boxed{3}$ $\quad 0 \underset{\text{ignored}}{\xrightarrow{a_{[q]}}} 1 \xrightarrow{d} 0 \underset{\text{ignored}}{\xrightarrow{a_{[q]}}} 1 \xrightarrow{a_{[q]}} 2 \xrightarrow{a_{[q]}} 3$

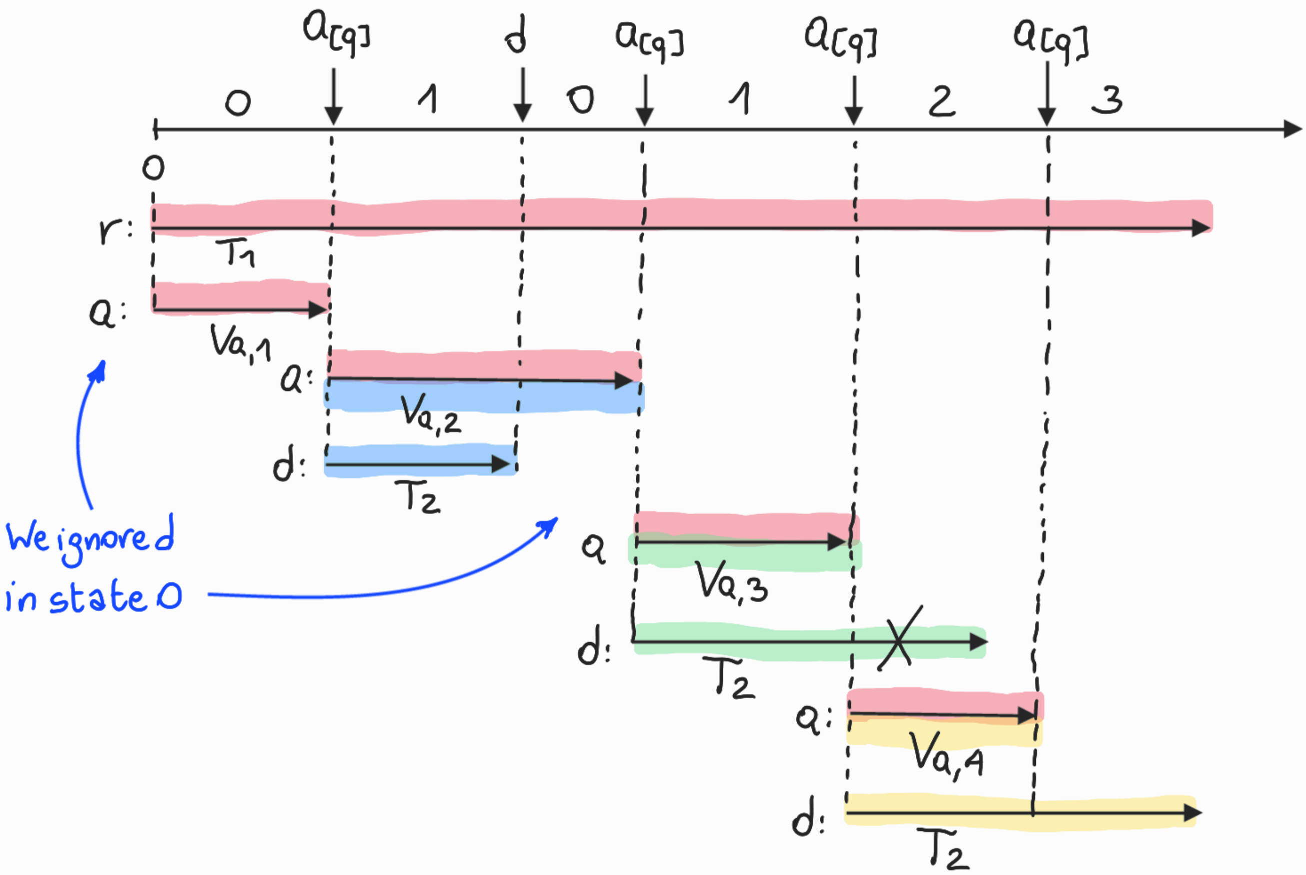
We compute the probabilities of the three cases using sample paths:

- $P(\boxed{1})$ was already computed in point 3) of this exercise
- For case $\boxed{2}$, the sample path is:



$$\Rightarrow P(\boxed{2}) = (1-q)q^3 \cdot P\left(V_{a,3} < T_2,\ V_{a,4} < T_2,\ V_{a,1} + V_{a,2} + V_{a,3} + V_{a,4} < T_1\right)$$

- For case $\boxed{3}$, the sample path is:



$$\Rightarrow P(\boxed{3}) = q^4 \cdot P\left(V_{a,2} > T_2,\ V_{a,3} < T_2,\ V_{a,4} < T_2,\ V_{a,1} + V_{a,2} + V_{a,3} + V_{a,4} < T_1\right)$$

Finally,

$$P(\ldots) = P(\boxed{1}) + P(\boxed{2}) + P(\boxed{3})$$