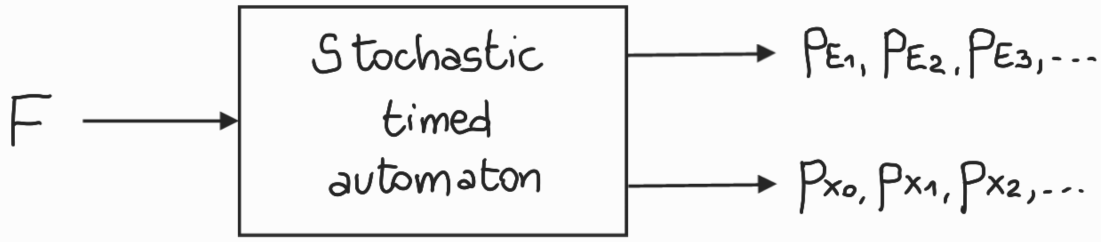
A stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, p_{X_0}, F)$ transforms the stochastic clock structure $F$ into the pmf's of the event and state sequences:




The basic brick to carry out this task, is the computation of probabilities of the type:

$$\boxed{P(E_k = e \mid X_{k-1} = x)}$$

Indeed, if these probabilities are known, then, using the total probability rule:

- $P(E_k = e) = \sum_{x \in \mathcal{X}} P(E_k = e \mid X_{k-1} = x) P(X_{k-1} = x)$
- $P(X_k = x') = \sum_{x \in \mathcal{X}} P(X_k = x' \mid X_{k-1} = x) P(X_{k-1} = x)$

$$P(X_k = x' \mid X_{k-1} = x) = \sum_{e \in \Gamma(x)} \underbrace{P(X_k = x' \mid X_{k-1} = x, E_k = e)}_{p(x' \mid x, e)} P(E_k = e \mid X_{k-1} = x)$$

$\Rightarrow$ Given $p_{X_0}$, we compute $p_{E_1}$ and $p_{X_1}$.

Then, given $p_{X_1}$, we compute $p_{E_2}$ and $p_{X_2}$.

And so on... $\Rightarrow$ iterative approach

Formally,

$$\boxed{P(E_k=e \mid X_{k-1}=x) = P\left(Y_{e,k-1} < \min_{\substack{e'\in\Gamma(x)\\ e'\neq e}} Y_{e',k-1}\right)}$$

Unfortunately, the distributions of the residual lifetimes are unknown in general, if we only assume the knowledge of the current state. Hence, $P(E_k=e \mid X_{k-1}=x)$ cannot be computed easily.

For this reason, to circumvent the problem, in the exercises we had to consider all the possible cases, starting from initialization, which results into a dramatic proliferation of cases as the event index k increases.

The question now is whether

there exist classes of stochastic timed automata for which computing $P(E_k=e \mid X_{k-1}=x)$ is an easy task.

The answer is given by stochastic timed automata with Poisson clock structure.

DEFINITION - A Poisson clock structure is a stochastic clock structure $F=\{F_e : e\in\mathcal{E}\}$ where all the distributions $F_e$ are exponential, i.e. for all $e\in\mathcal{E}$:

$$F_e(t) = \begin{cases} 1-e^{-\lambda_e t} & \text{if } t\geq 0 \\ 0 & \text{otherwise.} \end{cases} \qquad \lambda_e>0$$

## MAIN RESULT

For a stochastic timed automaton with Poisson clock structure, the residual lifetimes of an event have the same distribution as the corresponding total lifetimes.

In other words: $V_e \sim Exp\left(\frac{1}{\lambda_e}\right) \Rightarrow Y_e \sim Exp\left(\frac{1}{\lambda_e}\right)$

Sketch of the proof (by induction)

1) At initialization $(k=0)$:

$Y_{e,0} = V_{e,1}$ for all $e \in \Gamma(X_0)$

Since $V_{e,1} \sim Exp\left(\frac{1}{\lambda_e}\right)$, then $Y_{e,0} \sim Exp\left(\frac{1}{\lambda_e}\right)$

$\Rightarrow$ The statement is true for $k=0$.

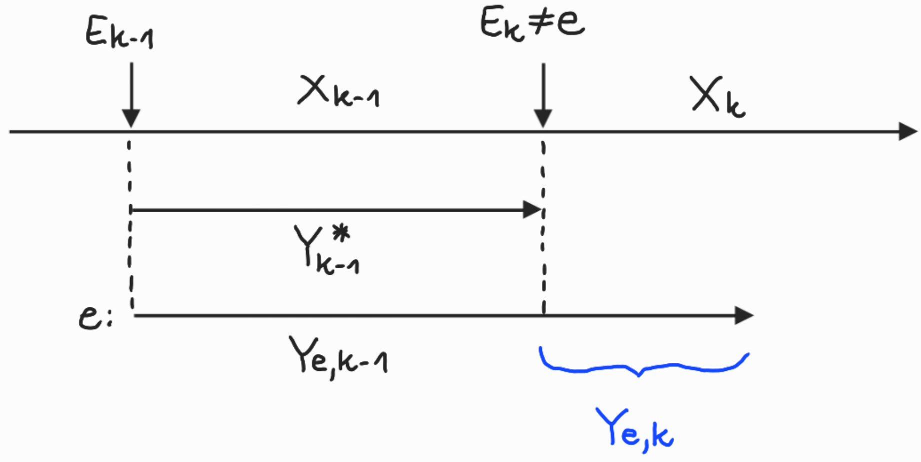
2) We assume that the statement is true for $k-1$, and we prove that it is true for $k$.

Case $i$) Event $e$ is activated in state $X_k$.

$\Rightarrow Y_{e,k} = V_{e,N_{e,k}}$

Since $V_{e,N_{e,k}} \sim Exp\left(\frac{1}{\lambda_e}\right)$, then $Y_{e,k} \sim Exp\left(\frac{1}{\lambda_e}\right)$

Case ii) Event $e$ "continues" in state $X_k$.




$$\Rightarrow \quad Y_{e,k} = Y_{e,k-1} - Y^*_{k-1}$$

$$P(Y_{e,k} \leq t \mid Y_{e,k-1} > Y^*_{k-1}) =$$

$$= P(Y_{e,k-1} - Y^*_{k-1} \leq t \mid Y_{e,k-1} > Y^*_{k-1})$$

$$= P(Y_{e,k-1} \leq t) = 1 - e^{-\lambda_e t}, \quad t \geq 0$$

Extended memoryless property

By induction

Hence, $Y_{e,k} \sim \mathrm{Exp}\left(\frac{1}{\lambda_e}\right)$.

$\Rightarrow$ The statement is true for $k$.

As a consequence of the main result, the following holds:

$$\boxed{P(E_k = e \mid X_{k-1} = x) = \frac{\lambda_e}{\Lambda(x)}}$$

where $\Lambda(x) \triangleq \sum_{e' \in \Gamma(x)} \lambda_{e'}$

We have a closed-form formula for $P(E_k = e \mid X_{k-1} = x)$!

# Proof

$$P(E_k = e \mid X_{k-1} = x)$$

$$= P\Big( Y_{e,k-1} < \min_{\substack{e' \in \Gamma(x) \\ e' \neq e}} Y_{e',k-1} \Big)$$

$\sim Exp\left(\frac{1}{\lambda_e}\right)$

Superposition

$\Rightarrow$ exp. distr. with rate $\sum_{\substack{e' \in \Gamma(x) \\ e' \neq e}} \lambda_{e'}$

$\sim Exp\left(\frac{1}{\lambda_{e'}}\right)$

$$= \frac{\lambda_e}{\lambda_e + \sum_{\substack{e' \in \Gamma(x) \\ e' \neq e}} \lambda_{e'}} = \frac{\lambda_e}{\Lambda(x)}$$

$$P(X<Y) = \frac{\lambda}{\lambda + \mu}$$

with $X \sim Exp\left(\frac{1}{\lambda}\right)$, $Y \sim Exp\left(\frac{1}{\mu}\right)$, independent

Exploiting the formula for $P(E_k = e \mid X_{k-1} = x)$ derived above, we can apply the iterative approach illustrated at the beginning for the computation of $p_{E_k}$ and $p_{X_k}$.

Let $\mathcal{E} = \{1, 2, \dots, m\}$ and $\mathcal{X} = \{1, 2, \dots, n\}$, and define the row vectors:

$$\pi_E(k) \triangleq \left[ P(E_k = 1) \;\; P(E_k = 2) \;\; \dots \;\; P(E_k = m) \right] \in \mathbb{R}^{1 \times m}$$

$$\pi_X(k) \triangleq \left[ P(X_k = 1) \;\; P(X_k = 2) \;\; \dots \;\; P(X_k = n) \right] \in \mathbb{R}^{1 \times n}$$

Moreover, define the matrices:

$$P_E \triangleq \begin{bmatrix} \frac{\lambda_1}{\Lambda(1)} & \frac{\lambda_2}{\Lambda(1)} & \cdots & \frac{\lambda_m}{\Lambda(1)} \\ \frac{\lambda_1}{\Lambda(2)} & \frac{\lambda_2}{\Lambda(2)} & \cdots & \frac{\lambda_m}{\Lambda(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_1}{\Lambda(n)} & \frac{\lambda_2}{\Lambda(n)} & \cdots & \frac{\lambda_m}{\Lambda(n)} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

and

$$P_X \triangleq \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,n} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n,1} & p_{n,2} & \cdots & p_{n,n} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

where $p_{i,j} \triangleq \sum_{e \in \Gamma(i)} p(j|i,e) \cdot \frac{\lambda_e}{\Lambda(i)}$.

Then we have:

$$P(E_k=j) = \sum_{i=1}^{n} \underbrace{P(E_k=j \mid X_{k-1}=i)}_{\frac{\lambda_j}{\Lambda(i)}} P(X_{k-1}=i)$$

j-th entry of $\Pi_E(k)$

entry in position $(i,j)$ of $P_E$

i-th entry of $\Pi_X(k-1)$

product of $\Pi_X(k-1)$ and the j-th column of $P_E$

$\Rightarrow$ In matrix form: $\boxed{\Pi_E(k) = \Pi_X(k-1)\, P_E}$

Similarly:

$$P(X_k=j) = \sum_{i=1}^{n} \underbrace{P(X_k=j \mid X_{k-1}=i)}_{p_{i,j}}\, P(X_{k-1}=i)$$

$j$-th entry of $\Pi_X(k)$

entry in position $(i,j)$ of $P_X$

$i$-th entry of $\Pi_X(k-1)$

product of $\Pi_X(k-1)$ and the $j$-th column of $P_X$

$\Rightarrow$ In matrix form: $\boxed{\Pi_X(k) = \Pi_X(k-1)\, P_X}$

Notice that:

$$\Pi_X(0) = [\,P(X_0=1)\ \ P(X_0=2)\ \dots\ P(X_0=n)\,]$$

$$= [\,p_{X_0}(1)\ \ p_{X_0}(2)\ \dots\ p_{X_0}(n)\,] \Rightarrow \underline{\text{known}}$$

Hence:

$$\Pi_X(1) = \Pi_X(0)\, P_X$$

$$\Pi_X(2) = \Pi_X(1)\, P_X = \Pi_X(0)\, P_X \cdot P_X = \Pi_X(0)\, P_X^2$$

$$\Pi_X(3) = \Pi_X(2)\, P_X = \Pi_X(0)\, P_X^2 \cdot P_X = \Pi_X(0)\, P_X^3$$

and so on.

We easily figure out that:

$$\boxed{\pi_X(k) = \pi_X(0) P_X^k \qquad k = 1, 2, 3, \dots}$$

Moreover, from $\pi_E(k) = \pi_X(k-1) P_E$:

$$\boxed{\pi_E(k) = \pi_X(0) P_X^{k-1} \cdot P_E \qquad k = 1, 2, 3, \dots}$$

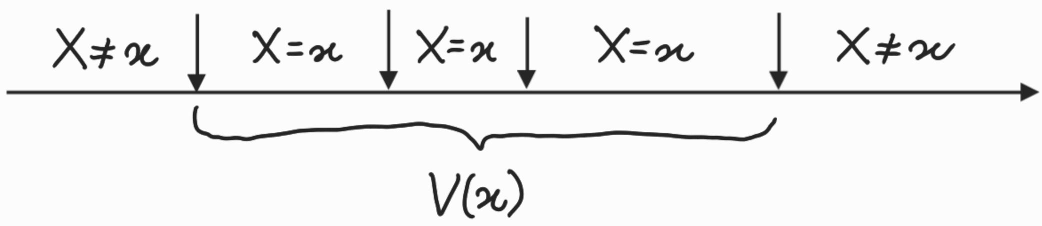
=> Computing $p_{E_k}$ and $p_{X_k}$ boils down to a problem of linear algebra

=> only matrix computations

=> computationally very efficient!

## STATE HOLDING TIME

DEFINITION - The state holding time $V(x)$ is the time that the system remains in state $x \in \mathcal{X}$.




(state holding time in state $x$)

We want to compute the distribution of $V(x)$ for a stochastic timed automaton with Poisson clock structure.

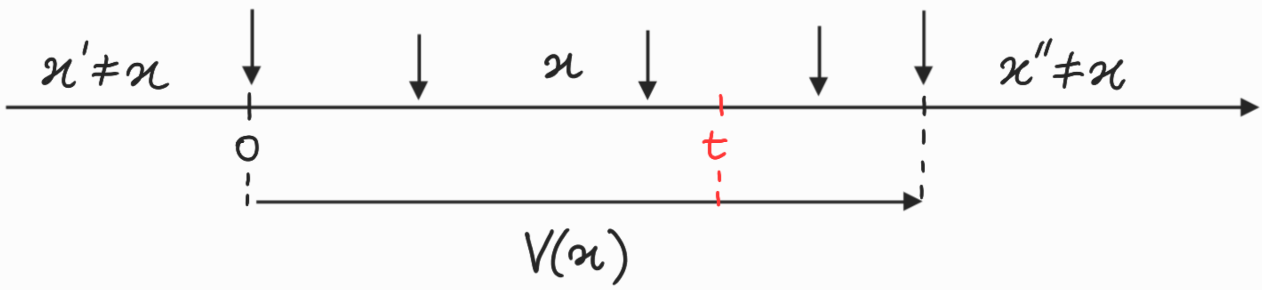
## PRELIMINARY RESULT

Let $X_1, X_2, \ldots, X_n, X_{n+1}$ be independent, identically distributed random variables with $X_i \sim Exp\left(\frac{1}{\lambda}\right)$ for all $i = 1, 2, \ldots, n+1$.

Then, for $t > 0$:

$$P(X_1 + \ldots + X_n < t,\ X_1 + \ldots + X_n + X_{n+1} > t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

Recalling that $P(V(x) \leq t) = 1 - P(V(x) > t)$, we start from computing $P(V(x) > t)$, for $t > 0$.




$$P(V(x) > t) =$$

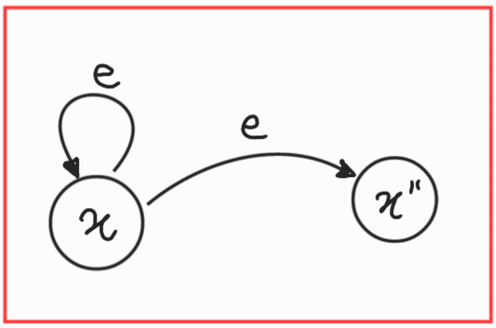
$$= P(\text{no state transition}^{(*)} \text{ occurs over } (0, t])$$

(*) we mean no transition to a state $x'' \neq x$

$$= P\left(\bigcap_{e \in \Gamma(x)} \{\text{no state transition triggered by event } e \text{ over } (0, t]\}\right)$$

$$\underset{\text{independent}}{=} \prod_{e \in \Gamma(x)} P(\text{no state transition triggered by event } e^{(\Delta)} \text{ over } (0, t])$$

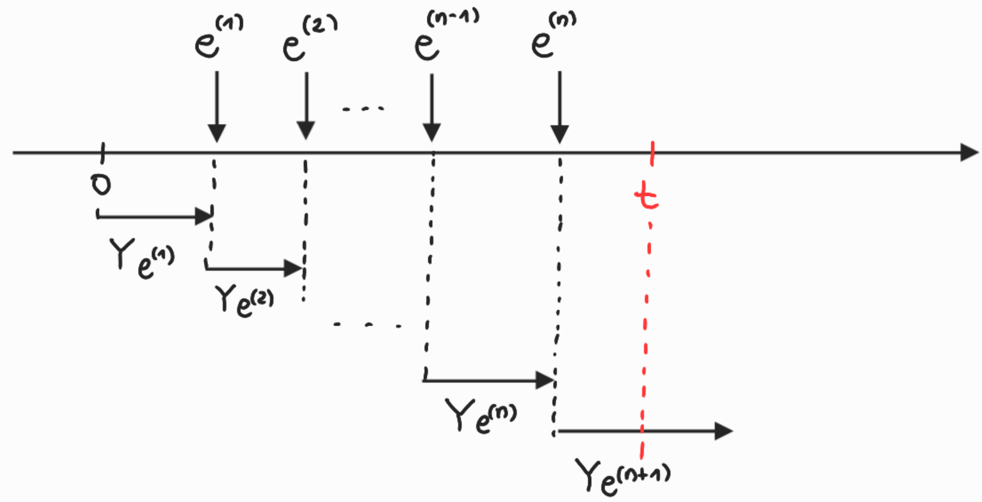
(Δ)



$$= \prod_{e\in\Gamma(x)} P\left(\bigcup_{n=0}^{\infty}\left\{\text{event } e \text{ occurs exactly } n \text{ times over } (0,t],\ \text{it does not trigger a state transition}\right\}\right)$$

$$\underset{\text{disjoint}}{=} \prod_{e\in\Gamma(x)} \sum_{n=0}^{\infty} P\left(\text{event } e \text{ occurs exactly } n \text{ times}^{(\square)} \text{ over } (0,t],\ \text{it does not trigger a state transition}\right)$$

(□)




where $e^{(i)}$ = $i$-th occurrence of event $e$, and $Y_{e^{(i)}}$ its lifetime

$\Rightarrow P(\text{event } e \text{ occurs exactly } n \text{ times over } (0,t])$

$$= P\left(Y_{e^{(1)}} + \dots + Y_{e^{(n)}} < t,\ Y_{e^{(1)}} + \dots + Y_{e^{(n)}} + Y_{e^{(n+1)}} > t\right)$$

$$= \frac{(\lambda_e t)^n}{n!} e^{-\lambda_e t}$$

$$= \prod_{e\in\Gamma(x)} \sum_{n=0}^{\infty} \underbrace{\frac{(\lambda_e t)^n}{n!} e^{-\lambda_e t}}_{\text{prob. event } e \text{ occurs } n \text{ times over } (0,t]} \cdot \underbrace{p(x|x,e)}_{\text{prob. to remain in state } x \text{ when event } e \text{ occurs}}{}^{n}$$

$n$ times
(Bernoulli scheme)

$$= \prod_{e\in\Gamma(x)} e^{-\lambda_e t} \overset{(0)}{\sum_{n=0}^{\infty}} \frac{[\lambda_e p(x|x,e)t]^n}{n!}$$

$$\overset{(0)}{} \boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$$= \prod_{e\in\Gamma(x)} e^{-\lambda_e t} \cdot e^{\lambda_e p(x|x,e)t}$$

$$= \prod_{e\in\Gamma(x)} e^{-\lambda_e [1-p(x|x,e)]t}$$

$$= e^{-\left\{\sum_{e\in\Gamma(x)} \lambda_e [1-p(x|x,e)]\right\}t}$$

$$\Rightarrow \boxed{P(V(x)\leq t) = 1 - e^{-\left\{\sum_{e\in\Gamma(x)} \lambda_e [1-p(x|x,e)]\right\}t}, \quad t\geq 0}$$

We have thus shown that $V(x)$ has an exponential distribution with rate $\sum_{e\in\Gamma(x)} \lambda_e [1-p(x|x,e)]$

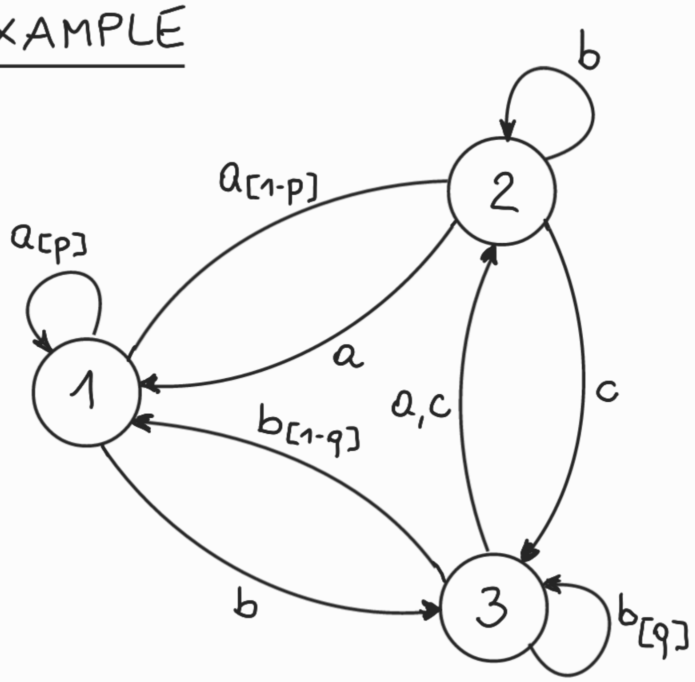
Notice that:

$$\sum_{x'\in\mathcal{X}} p(x'|x,e) = 1 \quad \Rightarrow \quad p(x|x,e) + \sum_{x'\neq x} p(x'|x,e) = 1$$

$$\Rightarrow \quad 1 - p(x|x,e) = \sum_{x'\neq x} p(x'|x,e)$$

$$\Rightarrow \boxed{\sum_{e\in\Gamma(x)} \lambda_e [1-p(x|x,e)] = \sum_{e\in\Gamma(x)} \sum_{x'\neq x} \lambda_e p(x'|x,e)}$$

# Example




| event | rate |
|---|---|
| a | $\lambda_a$ |
| b | $\lambda_b$ |
| c | $\lambda_c$ |

The rate of the exponential distribution of $V(1)$ is:

$$\lambda_a(1-p)+\lambda_b \quad \Rightarrow \quad E[V(1)] = \frac{1}{\lambda_a(1-p)+\lambda_b}$$

expected value

For $V(2)$:

$$\lambda_a+\lambda_c \quad \Rightarrow \quad E[V(2)] = \frac{1}{\lambda_a+\lambda_c}$$

For $V(3)$:

$$\lambda_a+\lambda_b(1-q)+\lambda_c \quad \Rightarrow \quad E[V(3)] = \frac{1}{\lambda_a+\lambda_b(1-q)+\lambda_c}$$

# Poisson processes

A Poisson process is the counting process of an event which is always possible, and characterized by interevent times that are:

- independent
- identically distributed with an <u>exponential distribution</u>.

The rate of the exponential distribution is also called the rate of the Poisson process.

A Poisson process can be viewed as a stochastic timed automaton with Poisson clock structure:

$$(\mathcal{E}, \mathcal{X}, \Gamma, f, x_0, F)$$

where $\mathcal{E} = \{e\}$, $\mathcal{X} = \{0, 1, 2, 3, \ldots\}$, the state transition diagram is given by

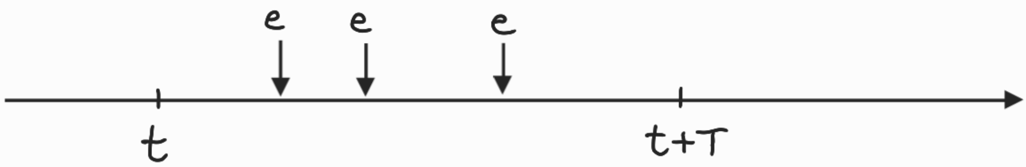
the state counts the number of occurrences of event e

$$\begin{tikzcd} {} \arrow[r] & 0 \arrow[r, bend left, "e"] & 1 \arrow[r, bend left, "e"] & 2 \arrow[r, bend left, "e"] & 3 \arrow[r, bend left, "e"] & \cdots \end{tikzcd}$$

and $F = \{F_e\}$ with $F_e(t) = 1 - e^{-\lambda t}$, $t \geq 0$, $\lambda > 0$.

Let

$N_e(t,t+T) \triangleq$ # occurrences of event $e$ over the interval $(t,t+T]$




$N_e(t,t+T)$ is a discrete random variable taking values in $\{0,1,2,\dots\}$. It can be shown that the pmf of $N_e(t,t+T)$ has the following form:

$$\boxed{P\left[N_e(t,t+T)=n\right] = \frac{(\lambda T)^n}{n!} e^{-\lambda T}, \quad n=0,1,2,\dots}$$

Poisson distribution with parameter $\lambda T$

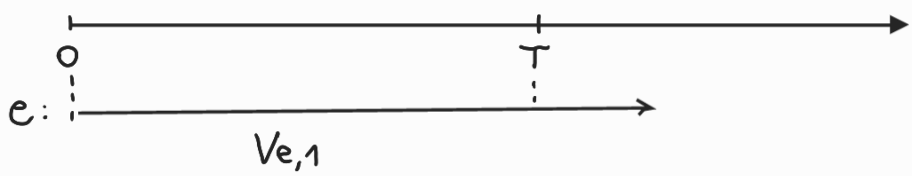
The expected value of $N_e(t,t+T)$ is

$$E\left[N_e(t,t+T)\right] = \lambda T.$$

Notice that the above formula does not depend on the time instant $t$, but only on the length $T$ of the interval.

$\Rightarrow$ It is a consequence of the memoryless property of the exponential distribution

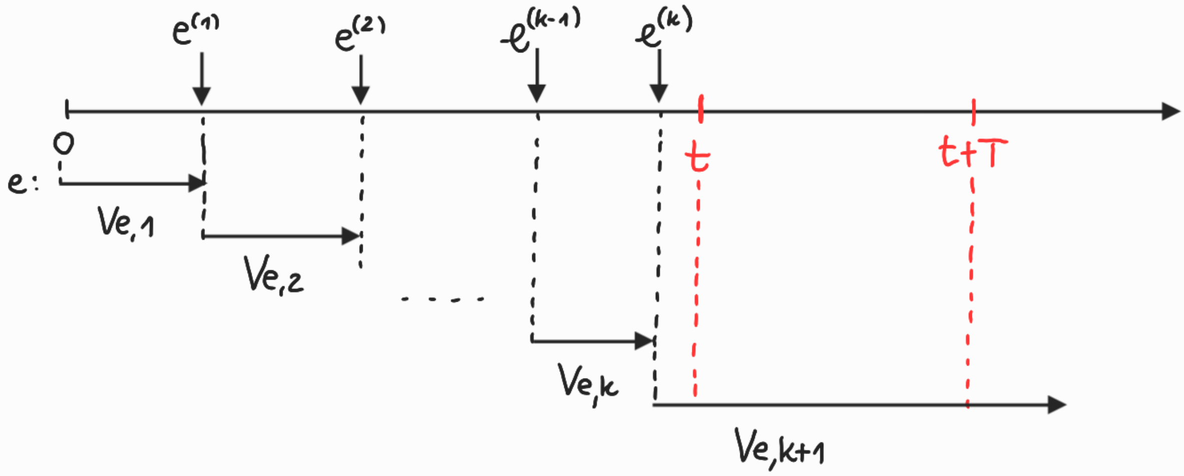
- Consider, for example, $P(N_e(0,T)=0)$.



$$\Rightarrow P(N_e(0,T)=0) = P(V_{e,1} > T) = 1 - P(V_{e,1} \leq T)$$
$$= 1 - (1 - e^{-\lambda T}) = e^{-\lambda T}$$

- Now consider $P(N_e(t,t+T)=0)$ for $t>0$.

  Say we had $k$ events $e$ before time $t$ ($k$ is unspecified).




$$\Rightarrow P(N_e(t,t+T)=0) =$$
$$= P\left(\sum_{h=1}^{k} V_{e,h} + V_{e,k+1} > t+T \,\middle|\, \sum_{h=1}^{k} V_{e,h} + V_{e,k+1} > t\right)$$
$$= P\left[V_{e,k+1} > \left(t - \sum_{h=1}^{k} V_{e,h}\right) + T \,\middle|\, V_{e,k+1} > t - \sum_{h=1}^{k} V_{e,h}\right]$$
$$= P(V_{e,k+1} > T) = 1 - P(V_{e,k+1} \leq T)$$
$$= 1 - (1 - e^{-\lambda T}) = e^{-\lambda T}$$

extended memoryless property

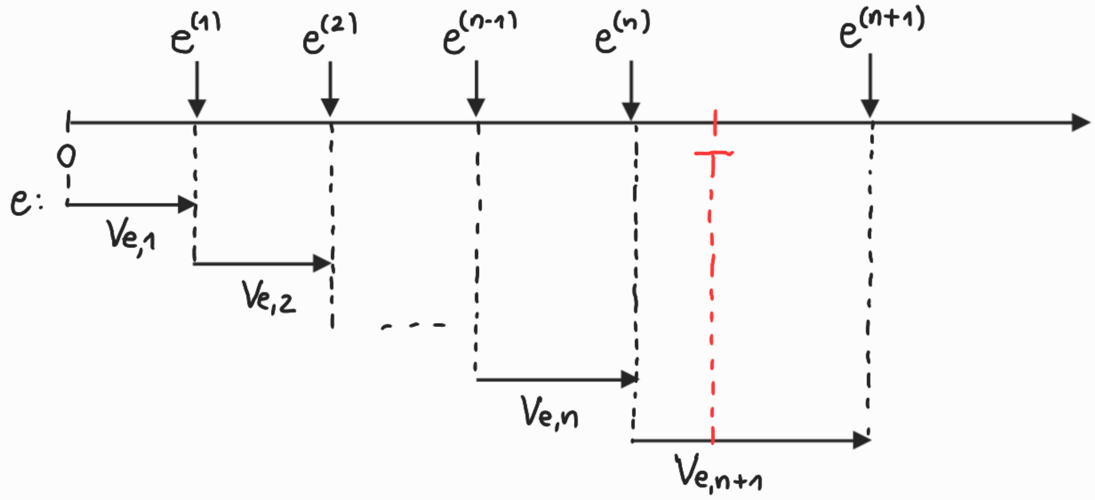
This random variable is positive, because it is conditional to the fact that k events e occurred before t

Hence, $P(N_e(t,t+T)=0) = e^{-\lambda T}$ independent of $t \geq 0$.

In the following, we will omit $t$ in the notation of $N_e$:

$$N_e(t,t+T) \longrightarrow N_e(T) \text{ independent of } t.$$

Notice that $P(N_e(T)=n)$ corresponds to the following sample path:




$$\Rightarrow \frac{(\lambda T)^n}{n!} = P(N_e(T)=n) = P\left(\sum_{i=1}^{n} V_{e,i} < T, \sum_{i=1}^{n+1} V_{e,i} > T\right)$$

We have thus proven the preliminary result on top of page 9. Simply pose $V_{e,i} \leftarrow X_i$, $i=1,\dots,n+1$, and $T \leftarrow t$.

**BYPRODUCT**

Let $X_1, X_2, \dots, X_n, X_{n+1}$ be independent, identically distributed random variables with $X_i \sim Exp\left(\frac{1}{\lambda}\right)$ for all $i=1,2,\dots,n+1$.

Then, for $t>0$:

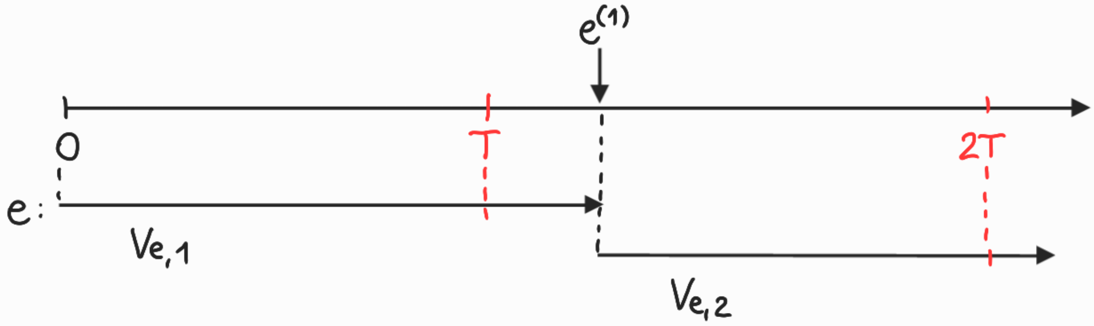
$$P(X_1+\dots+X_n < t, X_1+\dots+X_n+X_{n+1} > t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

## EXAMPLE (computations with Poisson processes)

Assume that event e is generated by a Poisson process with rate $\lambda > 0$.

- Compute the probability that event e does not occur over $(0, T]$, and occurs exactly once over $(T, 2T]$.

This probability corresponds to the following sample path:




$\Rightarrow P(\dots) =$

$P(T < V_{e,1} < 2T,\ V_{e,1} + V_{e,2} > 2T)$

using Poisson:

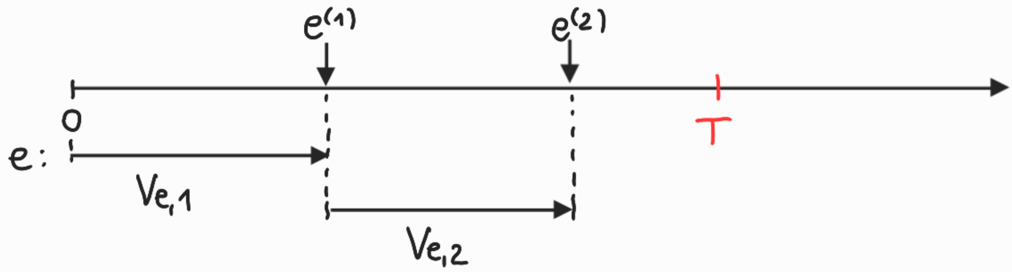
the two intervals are treated independently

$$P(N_e(0,T) = 0)\, P(N_e(T,2T) = 1)$$

$$= P(N_e(T) = 0)\, P(N_e(T) = 1) = e^{-\lambda T} \cdot (\lambda T) e^{-\lambda T}$$

- Compute the probability that event e occurs at least twice over $(0, T]$.

This probability corresponds to the following sample path:



Notice that, once we impose that the second event occurs before $T$, then the condition is satisfied, and we don't care about what happens next.

$$\Rightarrow P(\ldots) = \begin{cases} \nearrow P(V_{e,1}+V_{e,2}<T) \\ \searrow \text{using Poisson:} \end{cases}$$

$$P(N_e(T)\geq 2) = 1 - P(N_e(T)=0) - P(N_e(T)=1)$$

$$= 1 - e^{-\lambda T} - (\lambda T)e^{-\lambda T}$$

$$= 1 - e^{-\lambda T}(1+\lambda T)$$

BYPRODUCT

Let $X_1, X_2$ be independent, identically distributed random variables with $X_i \sim Exp\left(\frac{1}{\lambda}\right)$, $i=1,2$. Then, for $t>0$:

$$P(X_1+X_2<t) = 1 - e^{-\lambda t}(1+\lambda t)$$

Notice that, if the question was:

"Compute the probability that event e occurs exactly twice over $(0,T]$"

then the answer was:

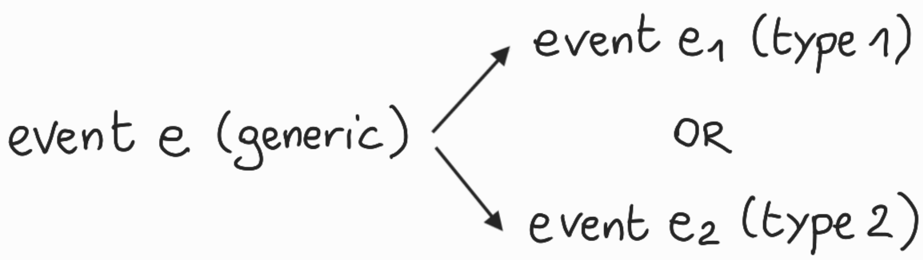
$$P(\ldots) = \begin{cases} \nearrow P(V_{e,1}+V_{e,2}<T,\ V_{e,1}+V_{e,2}+V_{e,3}>T) \\ \searrow \text{using Poisson: } P(N_e(T)=2) = \frac{(\lambda T)^2}{2}e^{-\lambda T} \end{cases}$$

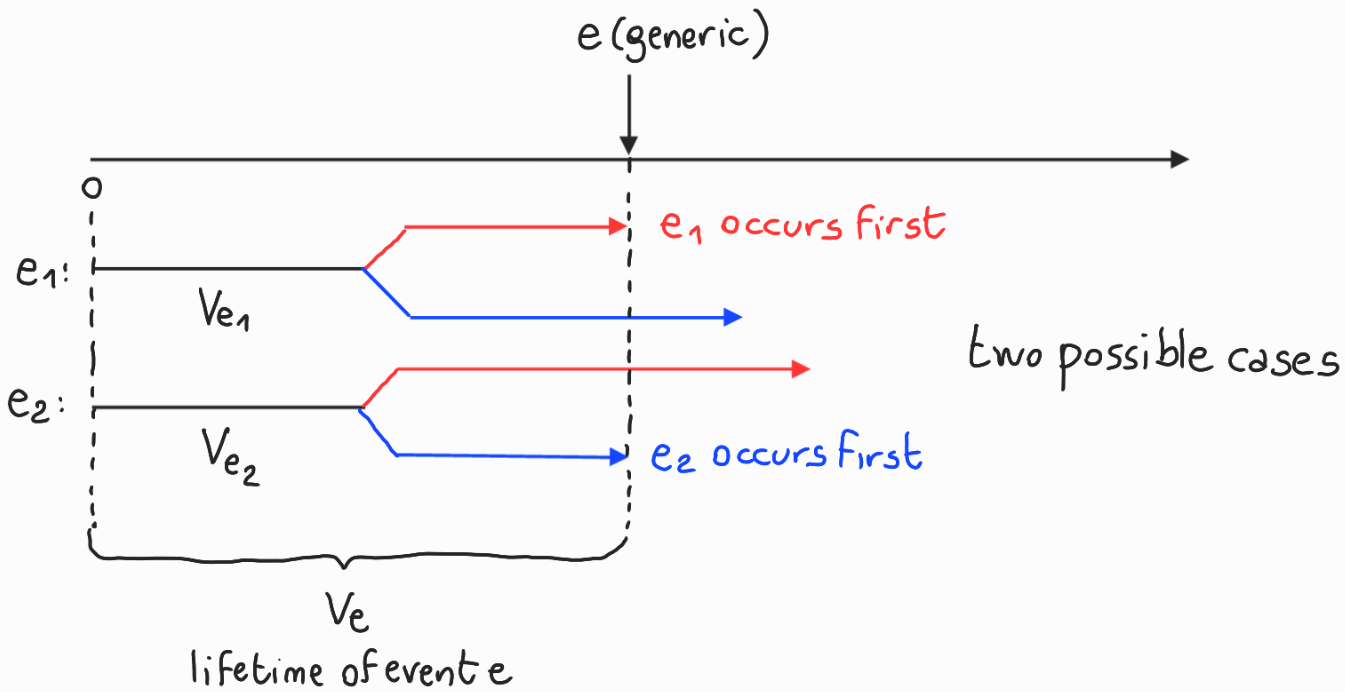
# Useful Result

Assume that event e is generated by a Poisson process with rate $\lambda > 0$.

Moreover, assume that, after its occurrence, event e is classified of type 1 with probability $q$, and of type 2 with probability $1-q$.

Then, the Poisson process generating event e can be viewed as the superposition of two independent Poisson processes, one with rate $\lambda q$ (generating events e classified of type 1), and the other with rate $\lambda(1-q)$ (generating events e classified of type 2).



$\Rightarrow$ Lifetime of event e: $V_e = \min\{V_{e,1}, V_{e,2}\}$

exp. distributed with rate $\lambda q$

exp. distributed with rate $\lambda(1-q)$

independent

superposition

$\Downarrow$

exp. distributed with rate

$\lambda q + \lambda(1-q) = \lambda$ ✓