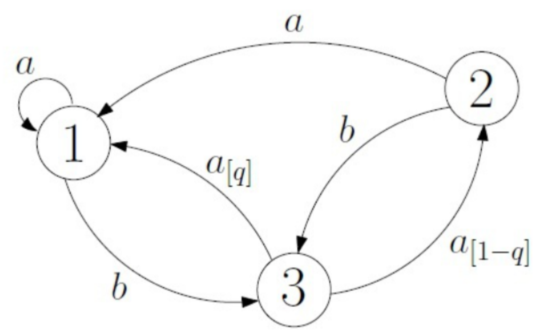
EXERCISE 1

Consider the stochastic timed automaton in the figure, where $q = 2/5$.



The initial state is uncertain, with pmf $p_{X_0}(1) = \frac{2}{3}$, $p_{X_0}(2) = 0$, and $p_{X_0}(3) = \frac{1}{3}$. Lifetimes of event a have a uniform distribution over the interval $[6, 9]$ min, while lifetimes of event b have an exponential distribution with expected value 5 min.

1. Compute $P(E_2 = a)$.
2. Compute $P(X_2 = 3)$.
3. Compute the probability that event b occurs at least once over the interval $[0, 10]$ min.
4. Compute the cdf of the state holding time in $x = 2$.

Stochastic clock structure

$V_a \sim U(6,9)$, $V_b \sim Exp(5) \rightarrow \frac{1}{\lambda} = 5 \Rightarrow \lambda = \frac{1}{5}$

All the lifetimes are expressed in minutes.

$$f_a(u) = \begin{cases} \frac{1}{3} & \text{if } 6 \leq u \leq 9 \\ 0 & \text{otherwise} \end{cases}$$ pdf of $V_{a,i}$

$$f_b(v) = \begin{cases} \frac{1}{5} e^{-\frac{v}{5}} & \text{if } v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$ pdf of $V_{b,i}$

1. Compute $P(E_2 = a)$.

First, we have to identify all the sample paths such that $\{E_2 = a\}$.
We can exclude sample paths starting with $\{X_0 = 2\}$, since $p_{X_0}(2) = 0$.

1) $X_0 = 1 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = a}$

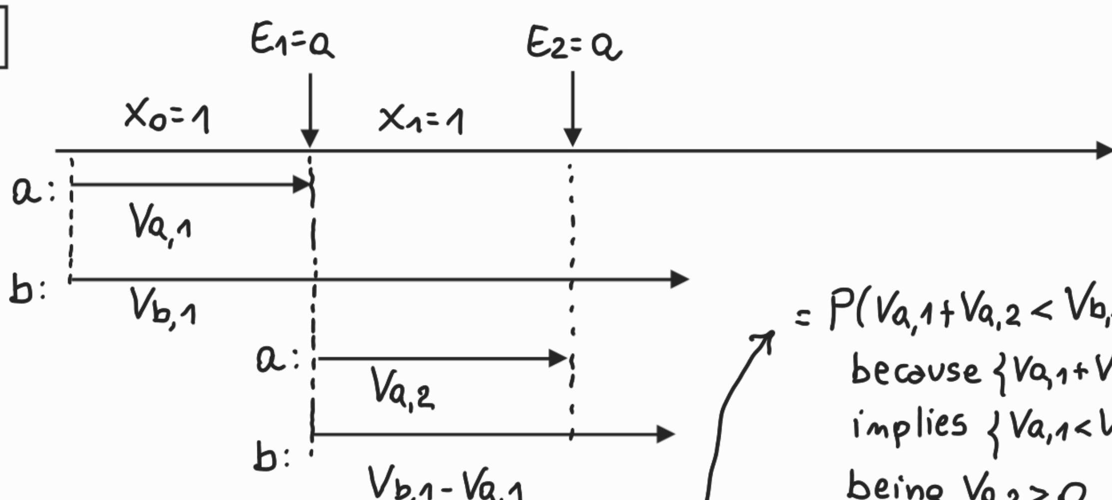
2) $X_0 = 1 \xrightarrow{E_1 = b} X_1 = 3 \xrightarrow{E_2 = a}$

3) $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = a}$

4) $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 2 \xrightarrow{E_2 = a}$

Then, we have to compute the probability of each sample path.

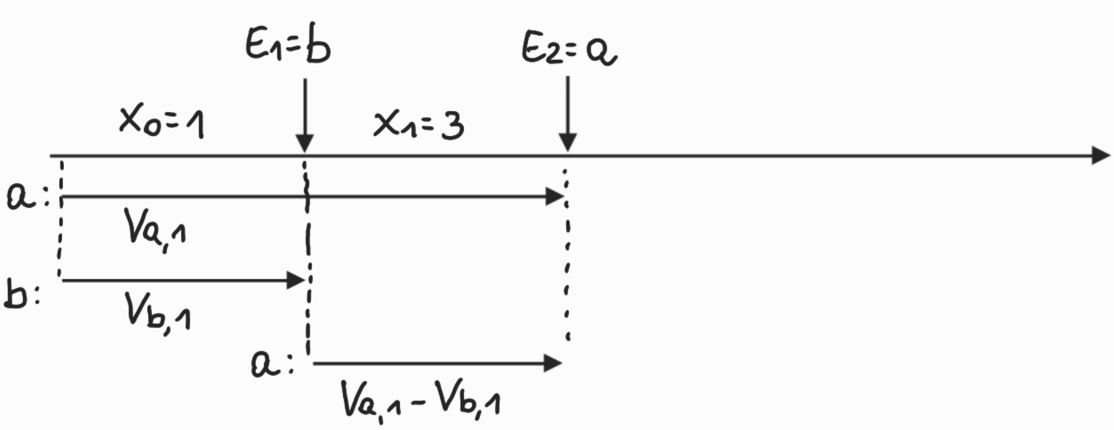
$\boxed{1}$



$= P(V_{a,1} + V_{a,2} < V_{b,1})$
because $\{V_{a,1} + V_{a,2} < V_{b,1}\}$
implies $\{V_{a,1} < V_{b,1}\}$,
being $V_{a,2} \geq 0$.

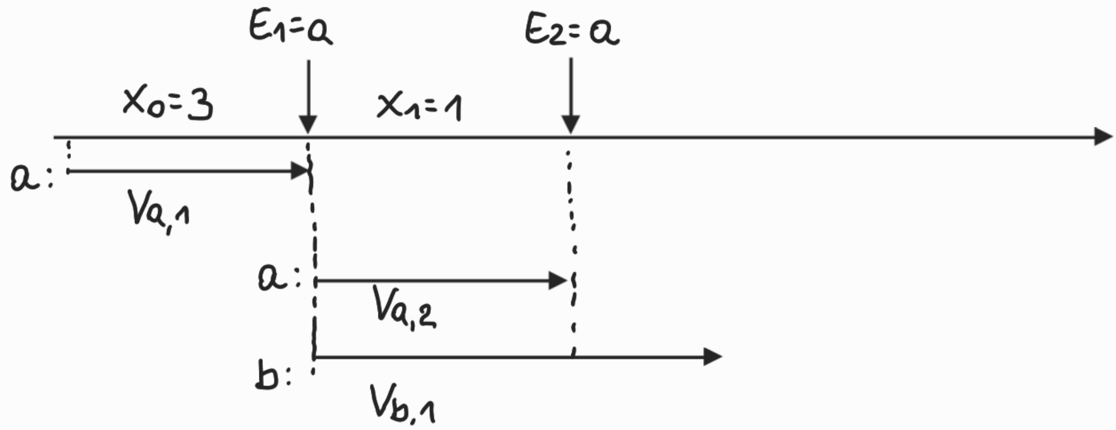
$$\Rightarrow P(\boxed{1}) = p_{X_0}(1) \cdot \underbrace{p(1|1,a)}_{1} \cdot P(V_{a,1} < V_{b,1},\ V_{a,2} < V_{b,1} - V_{a,1})$$

$\boxed{2}$



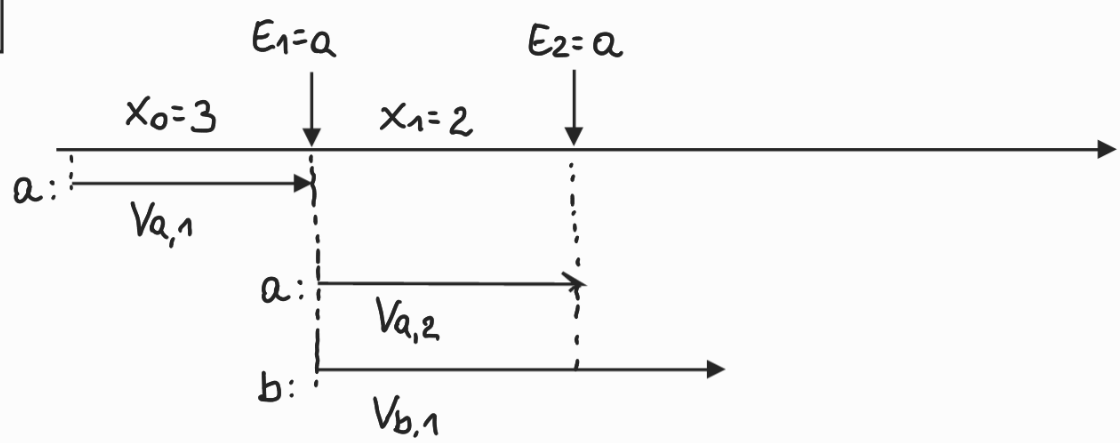
$$\Rightarrow P(\boxed{2}) = p_{x_0}(1)\cdot \underbrace{p(3|1,b)}_{1}\cdot P(V_{b,1} < V_{a,1})$$

$\boxed{3}$



$$\Rightarrow P(\boxed{3}) = p_{x_0}(3)\cdot \underbrace{p(1|3,a)}_{q}\cdot P(V_{a,2} < V_{b,1})$$

$\boxed{4}$



$$\Rightarrow P(\boxed{4}) = p_{x_0}(3)\cdot \underbrace{p(2|3,a)}_{1-q}\cdot P(V_{a,2} < V_{b,1})$$

Finally:

$$P(E_2 = a) = \sum_{i=1}^{4} P(\boxed{i})$$

$$= p_{x_0}(1)\, P(V_{a,1} + V_{a,2} < V_{b,1}) + p_{x_0}(1)\, P(V_{b,1} < V_{a,1}) + p_{x_0}(3)\, P(V_{a,2} < V_{b,1})$$

$\simeq 0.6254$ computed numericelly with Matlab

REMARK

Computing a probability involving two or more random variables requires the evaluation of a multiple integral.

In the exam this computation is typically <u>not requested.</u>

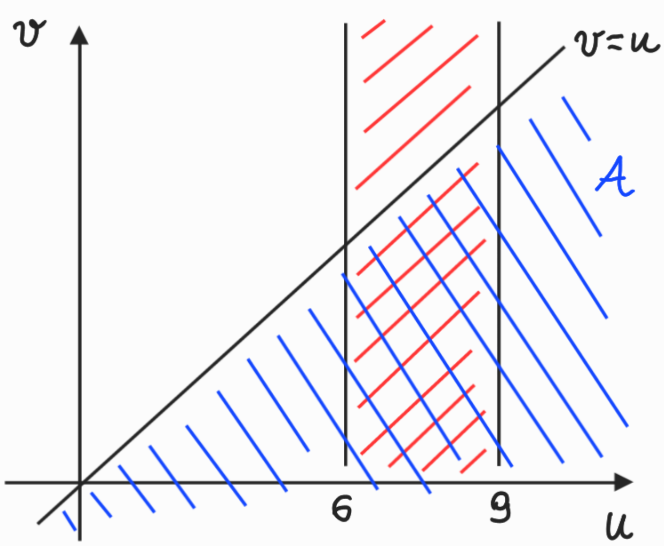
In the second part of the course we will see how to evaluate these probabilities numerically.

For illustrative purposes only, the computation of $P(V_{b,1} < V_{a,1})$ is shown next.

$$P(\underset{v}{V_{b,1}} < \underset{u}{V_{a,1}}) = \iint_A f_a(u) f_b(v)\,du\,dv$$

$V_{a,1}$ and $V_{b,1}$ are independent, therefore their joint pdf is the product of the marginal pdfs.

$A = \{(u,v) \in \mathbb{R}^2 : v < u\}$



The product $f_a(u) f_b(v)$ is nonzero only over the red region.

Hence, the integral is to be computed only over the blue-and-red region.

$$P(V_{b,1} < V_{a,1}) = \int_6^9 \int_0^u \frac{1}{3} \cdot \frac{1}{5} e^{-\frac{v}{5}}\,du\,dv = \int_6^9 \frac{1}{3}\left[-e^{-\frac{v}{5}}\right]_0^u du$$

$$= \int_6^9 \frac{1}{3}\left(1 - e^{-\frac{u}{5}}\right) du = \frac{1}{3}\left[u + 5e^{-\frac{u}{5}}\right]_6^9 = \frac{1}{3}\left(9 + 5e^{-\frac{9}{5}} - 6 - 5e^{-\frac{6}{5}}\right)$$

$$= 1 - \frac{5}{3}\left(e^{-\frac{6}{5}} - e^{-\frac{9}{5}}\right) \simeq 0.7735$$

2. Compute $P(X_2=3)$

First, we have to identify all the sample paths such that $\{X_2=3\}$.
As before, we can exclude sample paths starting with $\{X_0=2\}$.

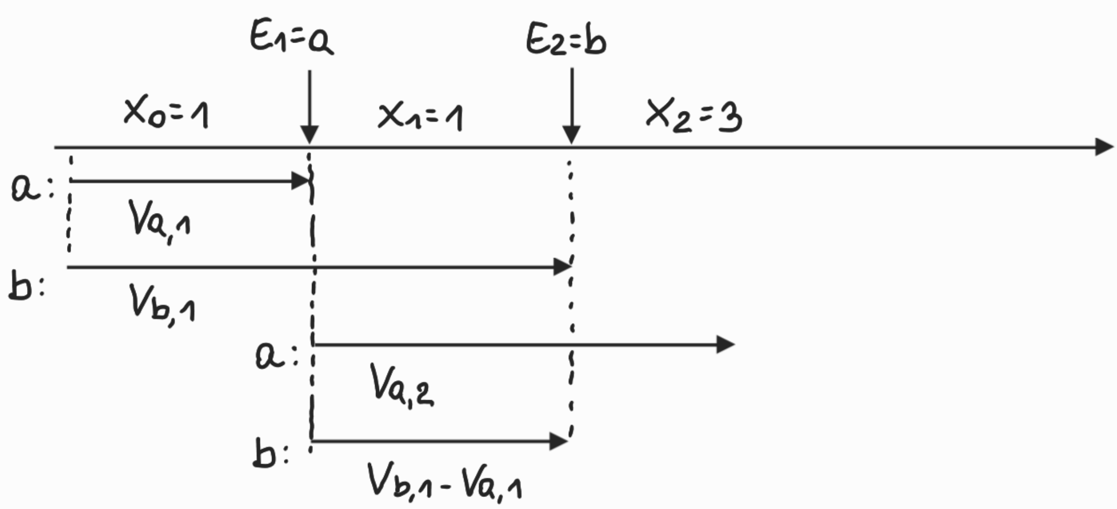
1) $X_0=1 \xrightarrow{E_1=a} X_1=1 \xrightarrow{E_2=b} X_2=3$

2) $X_0=3 \xrightarrow{E_1=a} X_1=1 \xrightarrow{E_2=b} X_2=3$

3) $X_0=3 \xrightarrow{E_1=a} X_1=2 \xrightarrow{E_2=b} X_2=3$

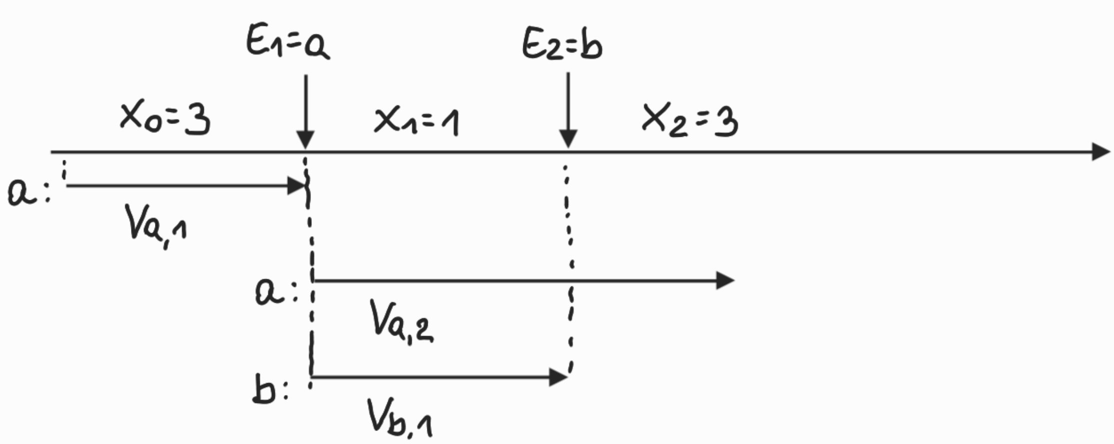
Then, we have to compute the probability of each sample path.

$\boxed{1}$



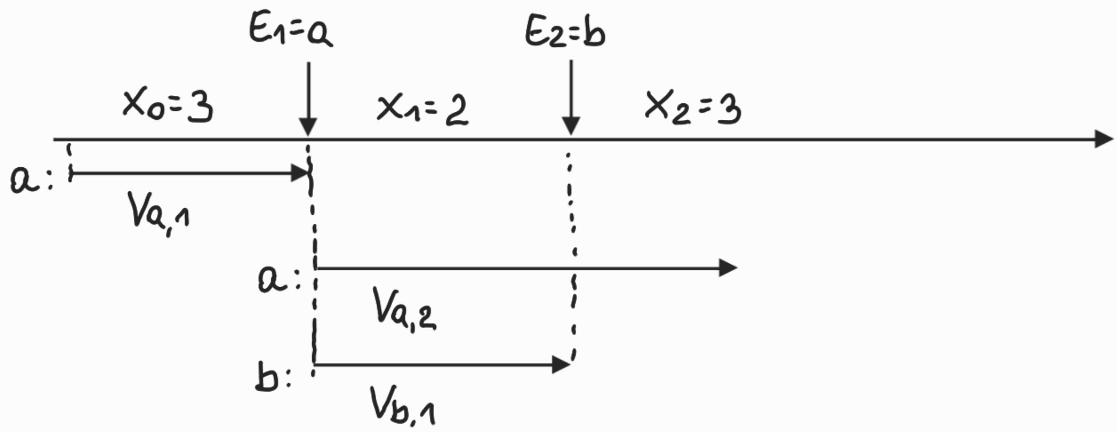
$$\Rightarrow P(\boxed{1}) = p_{X_0}(1)\cdot \underbrace{p(1|1,a)}_{1}\cdot \underbrace{p(3|1,b)}_{1}\cdot P(V_{a,1}<V_{b,1},\ V_{b,1}-V_{a,1}<V_{a,2})$$

$\boxed{2}$



$$\Rightarrow P(\boxed{2}) = p_{x_0}(3) \cdot \underbrace{p(1|3,a)}_{q} \cdot \underbrace{p(3|1,b)}_{1} \cdot P(V_{b,1} < V_{a,2})$$

$\boxed{3}$



$$\Rightarrow P(\boxed{3}) = p_{x_0}(3) \cdot \underbrace{p(2|3,a)}_{1-q} \cdot \underbrace{p(3|2,b)}_{1} \cdot P(V_{b,1} < V_{a,2})$$

Finally:

$$P(X_2=3) = \sum_{i=1}^{3} P(\boxed{i})$$

$$= p_{x_0}(1) P(V_{a,1} < V_{b,1} < V_{a,1} + V_{a,2}) + p_{x_0}(3) P(V_{b,1} < V_{a,2})$$

$$\simeq 0.3746$$

computed numerically with Matlab

3. Compute the probability that event b occurs at least once over the interval [0, 10] min.

If b were always possible, the answer would be trivial:

$$P(V_{b,1} \leq T) = F_b(T) = 1 - e^{-T/5}, \quad \text{where } T = 10 \text{ min.}$$

Unfortunately, this is not the case, because event b is not possible in state 3.

On the other hand, event a is always possible.

Therefore, we have in principle to consider all the possible sample paths with event b preceded by an undetermined number of events a, while imposing that event b occurs before time T.

Luckily enough, since $V_a \sim U(6,9)$, we can exploit the fact that event a may occur at most once before time $T = 10$ min (the second occurrence cannot be before $6+6 = 12$ min).

The possible cases are thus the following:

$\boxed{1}$ $X_0 = 1 \xrightarrow{E_1 = b}$

$\boxed{2}$ $X_0 = 1 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = b}$

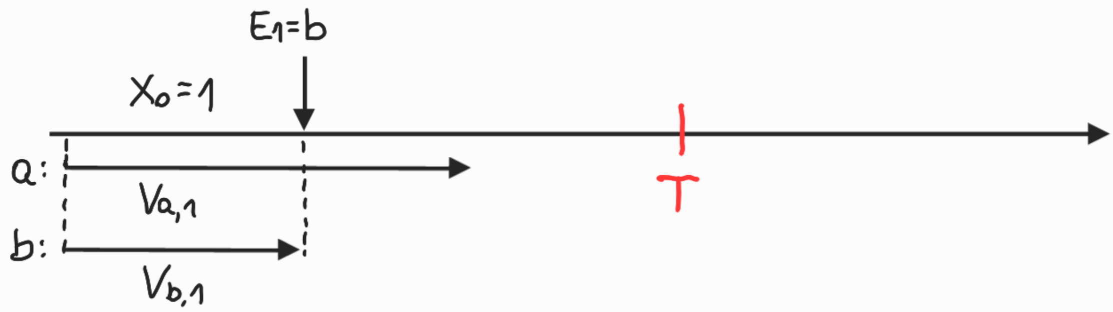
$\boxed{3}$ $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = b}$

$\boxed{4}$ $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 2 \xrightarrow{E_2 = b}$

+ impose that event b occurs before time T

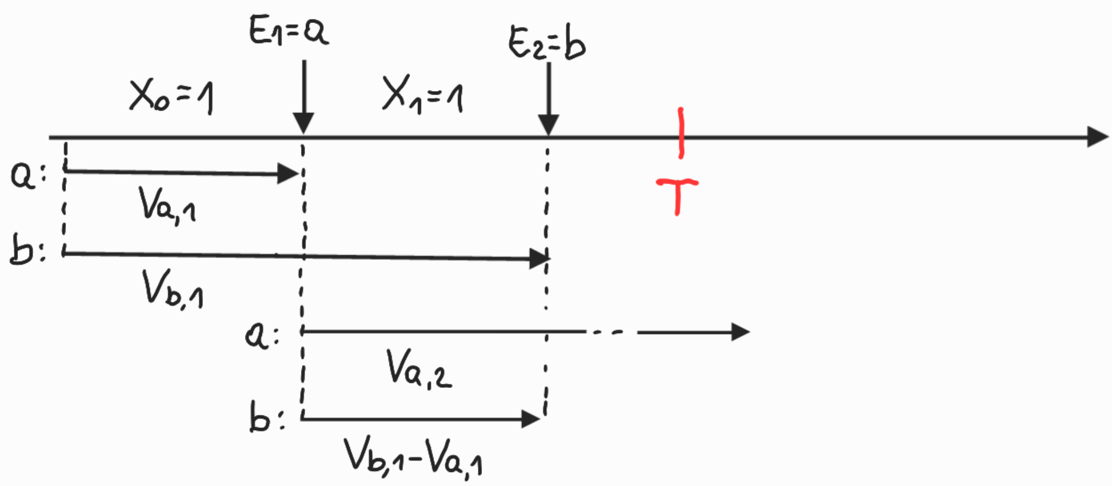
We compute the probability of each case.

$\boxed{1}$



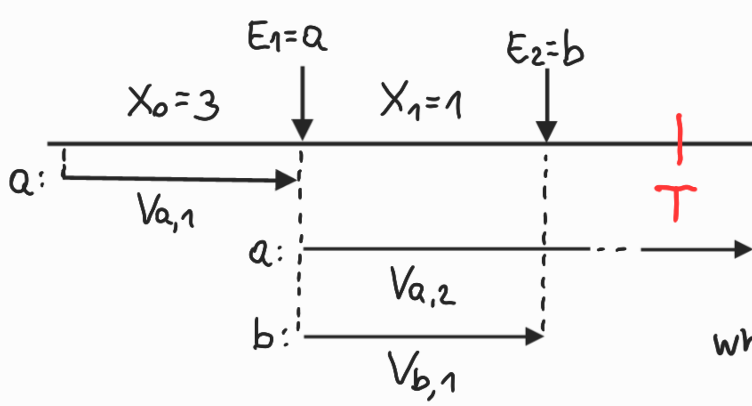
$$\Rightarrow \quad P(\boxed{1}) = p_{X_0}(1)\, P\left(V_{b,1} < V_{a,1},\ V_{b,1} < T\right)$$

$\boxed{2}$



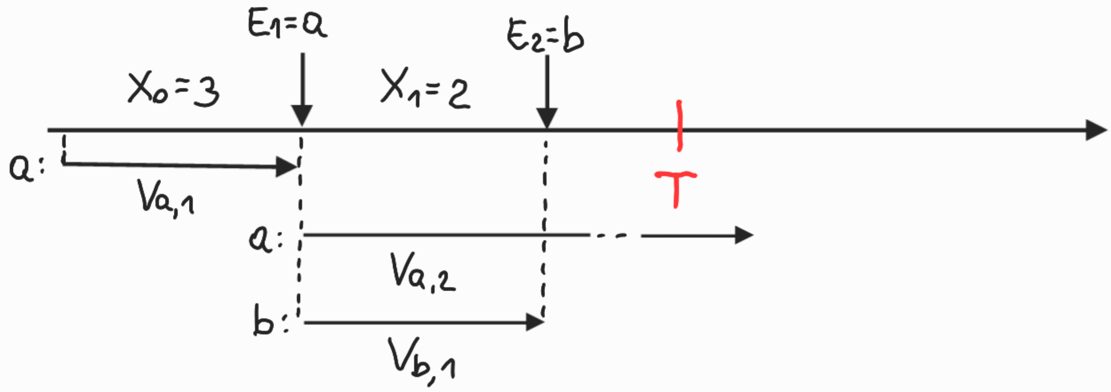
$$\Rightarrow \quad P(\boxed{2}) = p_{X_0}(1)\, P\left(V_{a,1} < V_{b,1} < V_{a,1} + V_{a,2},\ V_{b,1} < T\right)$$

$\boxed{3}$



$$\Rightarrow \quad P(\boxed{3}) = p_{X_0}(3) \cdot q \cdot P\left(V_{b,1} < V_{a,2},\ \overbrace{V_{a,1} + V_{b,1}}^{\text{time instant when event b occurs}} < T\right)$$

$\boxed{4}$



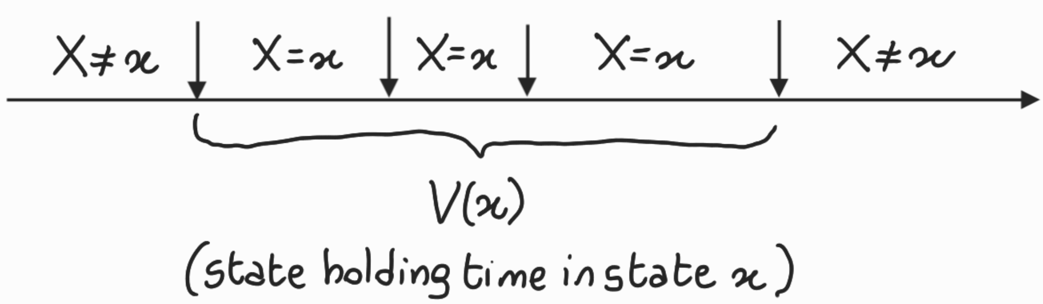
$$P(\boxed{4}) = p_{X_0}(3) \cdot (1-q) \cdot P(V_{b,1} < V_{a,2},\ V_{a,1} + V_{b,1} < T)$$

Finally,

$$P(\text{event } b \text{ occurs before } T=10) = \sum_{i=1}^{4} P(\boxed{i})$$

$\simeq 0.7046$ ← computed numerically with Matlab

4. The state holding time is the time that the system remains in a given state.



We have to compute the cdf of $V(2)$.

We observe that:

- the system enters state 2 only from state 3 with event a, and in state 3 event b is not possible
  $\Rightarrow$ in state 2 the lifetimes of both events are total lifetimes.

- the system leaves state 2 when either of the two events occurs.

Hence, we have:

$$V(2) = \min\{\underbrace{V_a}_{\substack{\text{total}\\\text{lifetime}\\\text{of event a}}}, \underbrace{V_b}_{\substack{\text{total}\\\text{lifetime}\\\text{of event b}}}\}$$

To compute the cdf of $V(2)$, we start by computing

$$P(V(2)>t) = P(\min\{V_a, V_b\}>t)$$

$$\underset{\{\min\{V_a,V_b\}>t\} \Leftrightarrow \{V_a>t, V_b>t\}}{=} P(V_a>t, V_b>t) \underset{\text{independent}}{=} P(V_a>t)P(V_b>t)$$

$$= [1-F_a(t)][1-F_b(t)]$$

$$\Rightarrow P(V(2)\leq t) = 1-P(V(2)>t) = 1-[1-F_a(t)][1-F_b(t)]$$

Since

$$F_a(t) = \begin{cases} 0 & \text{if } t<6 \\ \frac{t-6}{3} & \text{if } 6\leq t\leq 9 \\ 1 & \text{if } t>9 \end{cases} \qquad F_b(t) = \begin{cases} 1-e^{-\frac{t}{5}} & \text{if } t\geq 0 \\ 0 & \text{if } t<0 \end{cases}$$

we have:

$$P(V(2)\leq t) = \begin{cases} 0 & \text{if } t<0 \\ 1-e^{-\frac{t}{5}} & \text{if } 0\leq t<6 \\ 1-\frac{9-t}{3}e^{-t/5} & \text{if } 6\leq t\leq 9 \\ 1 & \text{if } t>9 \end{cases}$$

REMARK

In state 1 we have:

$$V(1) = Y_b$$ residual lifetime of event b

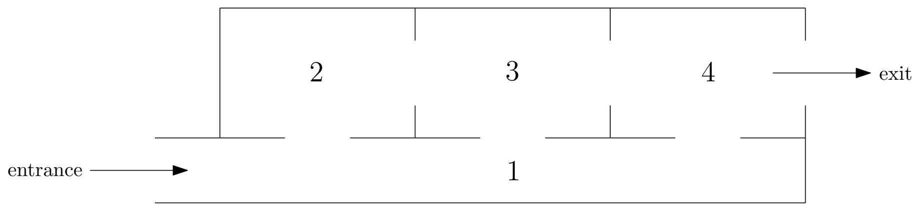
because the system leaves state 1 only with event b, but when the system enters state 1, event b may have a residual lifetime from the previous state. Indeed,

- $Y_b = V_b - V_a$ if the system arrives from state 2,
- $Y_b = V_b$ if the system arrives from state 3.

Computing $P(V(1) \leq t)$ is therefore a non-trivial task...

Exercise 2

The figure represents the map of an office composed of four rooms, numbered from 1 to 4.



The time spent by an employee in a room has a uniform distribution between 16 and 30 min. When the employee leaves a room, she takes one of the alternatives with equal probability.

1. Model the presence of the employee in the office using a stochastic timed automaton, starting from when she enters the office at 8 AM.
2. Compute the probability that the employee visits all the rooms, before returning to any room for the second time.
3. Compute again the previous probability, adding the constraint that all the rooms are visited before 9 AM.
4. Compute the probability that the employee exits the office before 9 AM.

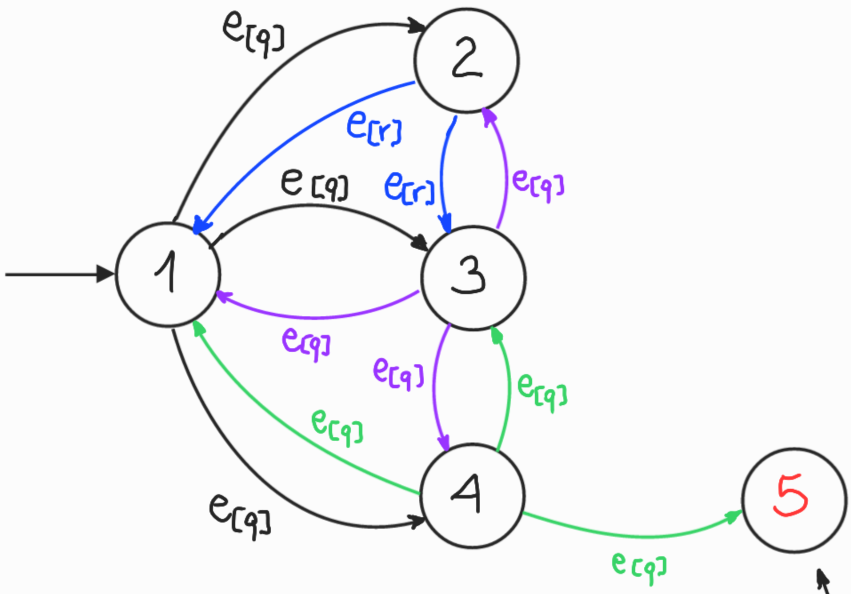
1. Stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$

Events $\mathcal{E} = \{e\}$

The employee leaves the current room

State $x = \#$room where the employee is $\in \{1, 2, 3, 4, 5\}$

This value represents the fact that the employee is no more in the office



$$q = \frac{1}{3}$$

$$r = \frac{1}{2}$$

Notice that there are no events in state 5 (≡ terminal state)

Stochastic clock structure: $F = \{F_e\}$

$$V_e \sim U(16, 30) \implies F_e(t) = P(V_e \leq t) = \begin{cases} 0 & \text{if } t < 16 \\ \dfrac{t-16}{14} & \text{if } 16 \leq t \leq 30 \\ 1 & \text{if } t > 30 \end{cases}$$

The lifetimes of event e coincide with state holding times

2. We first identify all the possible cases:

$\boxed{1}$ $\quad 1 \xrightarrow{e[q]} 2 \xrightarrow{e[r]} 3 \xrightarrow{e[q]} 4$

$\boxed{2}$ $\quad 1 \xrightarrow{e[q]} 4 \xrightarrow{e[q]} 3 \xrightarrow{e[q]} 2$

Second, we compute the probability of each case:

$$P(\boxed{1}) = q \cdot r \cdot q = q^2 r$$

$$P(\boxed{2}) = q \cdot q \cdot q = q^3$$

$$\Rightarrow P(\dots) = P(\boxed{1}) + P(\boxed{2}) = q^2(q+r) \simeq 0.0926$$

3. Let $T = 60$ min (time interval between 8 AM and 9 AM).

The possible cases are the same as before:

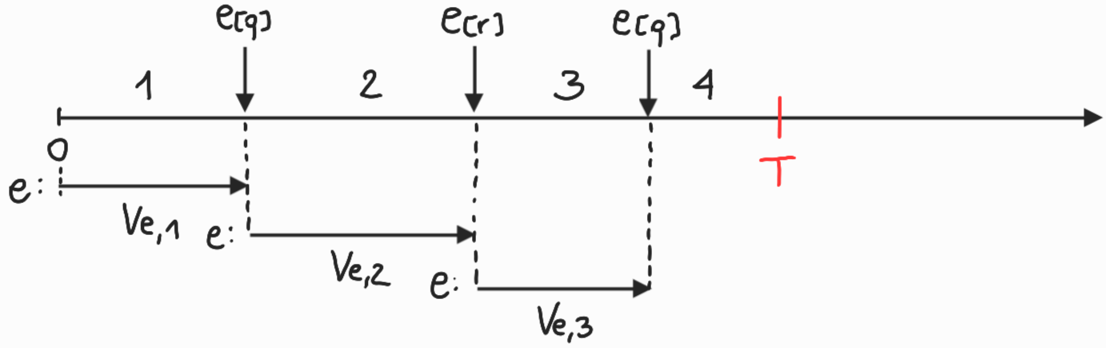
$\boxed{1}$ $\quad 1 \xrightarrow{e[q]} 2 \xrightarrow{e[r]} 3 \xrightarrow{e[q]} 4$

$\boxed{2}$ $\quad 1 \xrightarrow{e[q]} 4 \xrightarrow{e[q]} 3 \xrightarrow{e[q]} 2$

+ add the constraint that these paths are completed before T

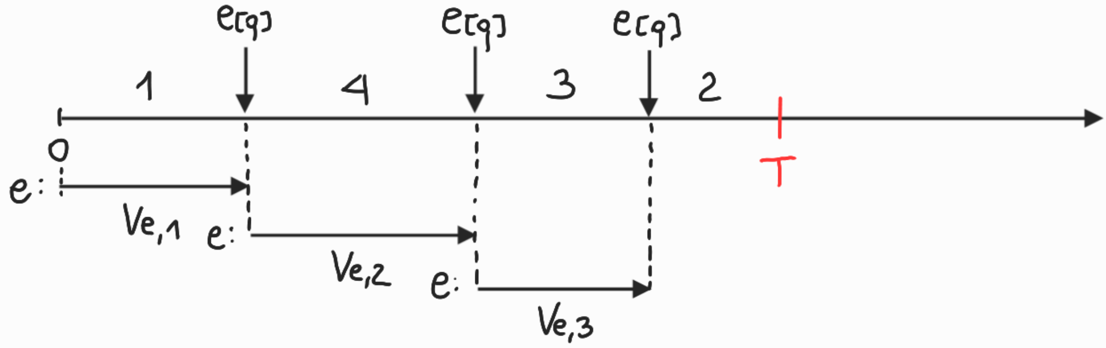
We compute the probability of each case, using sample paths.

$\boxed{1}$



$$P(\boxed{1}) = q \cdot r \cdot q \cdot P(V_{e,1} + V_{e,2} + V_{e,3} < T)$$

$\boxed{2}$



$$P(\boxed{2}) = q \cdot q \cdot q \cdot P\left(V_{e,1} + V_{e,2} + V_{e,3} < T\right)$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) = q^2(q+r) \cdot \underbrace{P\left(V_{e,1} + V_{e,2} + V_{e,3} < T\right)}_{\substack{\text{estimated using Matlab} \\ \simeq 0.1050}}$$

$$\simeq 0.097$$

4. We notice that, since the time spent in each room has a uniform distribution over [16,30] min, the employee can visit <u>at most three rooms</u>, and then exit before 9AM.

The possible cases are thus the following, where $T = 60$ min:

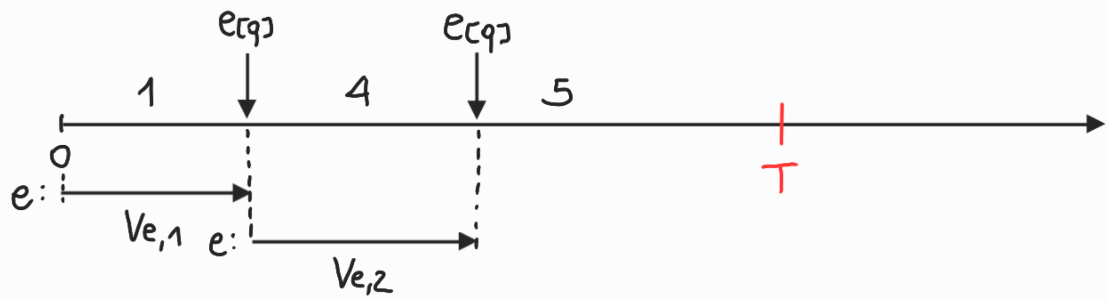
$\boxed{1}$ $\quad 1 \xrightarrow{e[q]} 4 \xrightarrow{e[q]} 5$

$\boxed{2}$ $\quad 1 \xrightarrow{e[q]} 3 \xrightarrow{e[q]} 4 \xrightarrow{e[q]} 5$

+ add the constraint that these paths are completed before T

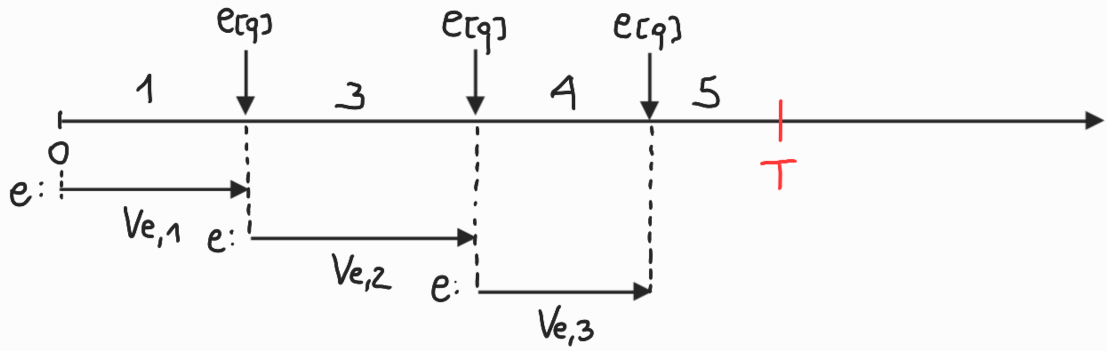
We compute the probability of each case, using sample paths.

$\boxed{1}$



$$P(\boxed{1}) = q \cdot q \cdot \underbrace{P\left(V_{e,1} + V_{e,2} < T\right)}_{= 1} = q^2$$

$\boxed{2}$



$$P(\boxed{2}) = q \cdot q \cdot q \cdot P(V_{e,1} + V_{e,2} + V_{e,3} < T)$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) = q^2 + q^3 \cdot P(V_{e,1} + V_{e,2} + V_{e,3} < T)$$

$$\simeq 0.1150$$